Math 1131 Review

Dec. 8, 2023

Practice problem 1

A function $f(x)$ satisfies $f''(x) = 2 - 3x$ with $f'(0) = -1$ and $f(0) = 1$. Compute $f(2)$.

$$f''(x) = 2 - 3x \implies f'(x) = 2x - \frac{3x^2}{2} + C$$

Set $x=0$: $f'(0) = C$

$-1 = C$

$$f'(x) = 2x - \frac{3x^2}{2} - 1$$

$$\rightsquigarrow f(x) = x^2 - \frac{\cancel{3}}{2} \cdot \frac{x^3}{\cancel{3}} - x + \tilde{C}$$

Set $x=0$: $f(0) = \tilde{C} \implies 1 = \tilde{C}$, so

$$f(x) = x^2 - \frac{x^3}{2} - x + 1 : f(2) = 2^2 - \frac{2^3}{2} - 2 + 1$$

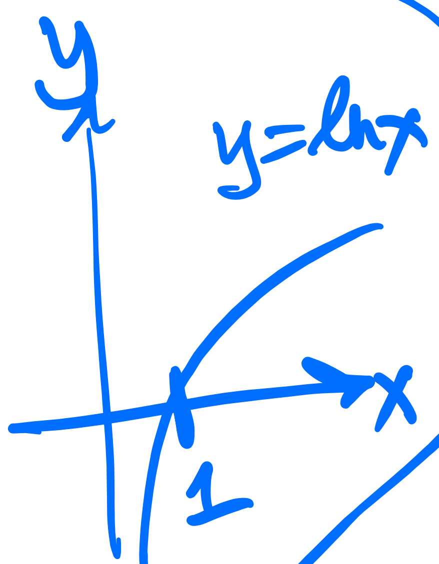
$$= \cancel{4} - \cancel{4} - 2 + 1 = -1$$

Find $f(x)$ if $f'(x) = 3x^2 + \dfrac{2}{x}$ for $x > 0$ and $f(1) = 3$.

$$f'(x) = 3x^2 + \frac{2}{x} \Rightarrow f(x) = \int \left(3x^2 + \frac{2}{x}\right) dx$$

$$= 3\int x^2\,dx + 2\int \frac{1}{x}\,dx$$

$$f(x) = \cancel{3}\frac{x^3}{\cancel{3}} + 2\ln x + C$$



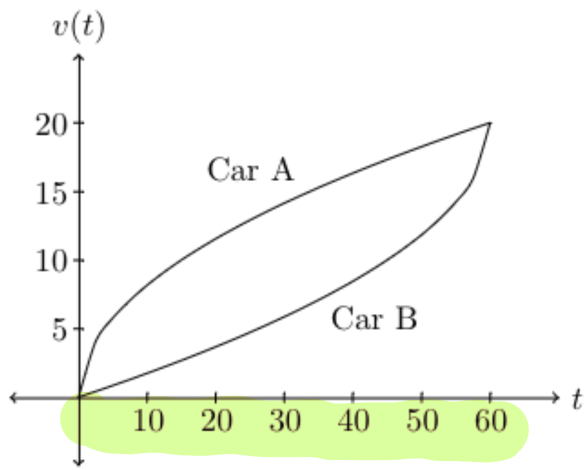
Set $x=1$: $f(1) = 1 + 2\cancel{\ln(1)}^{\,0} + C$

$$3 = 1 + C$$

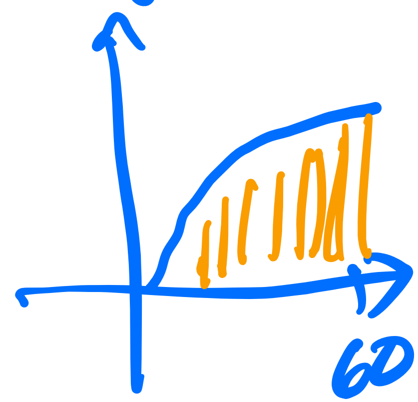
$$\Rightarrow C = 3 - 1 = 2$$

$$\boxed{f(x) = x^3 + 2\ln x + 2}$$

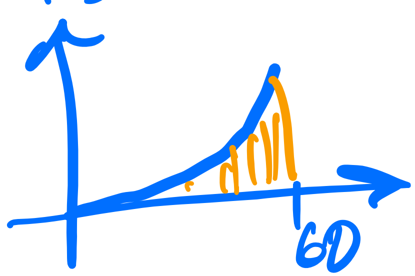
Below is the graph of the velocity (in ft/sec) over the interval $0 \le t \le 60$ for Car A and Car B. ~~How do their distances traveled compare over this interval?~~ Which car travels farther?



Distance traveled by Car A is area under



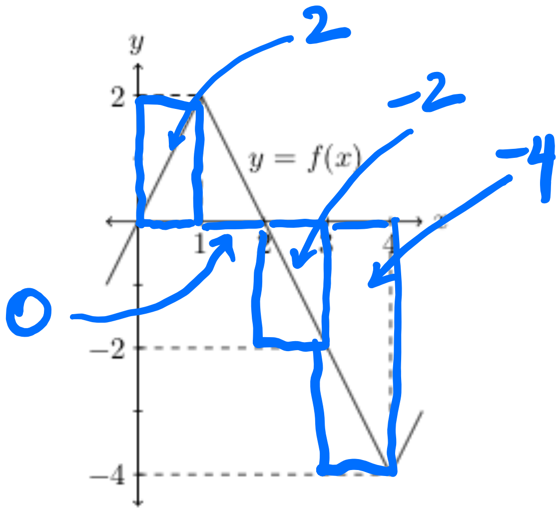
Distance traveled by Car B is area under



The A-area is bigger than B-area, so Car A traveled a longer distance

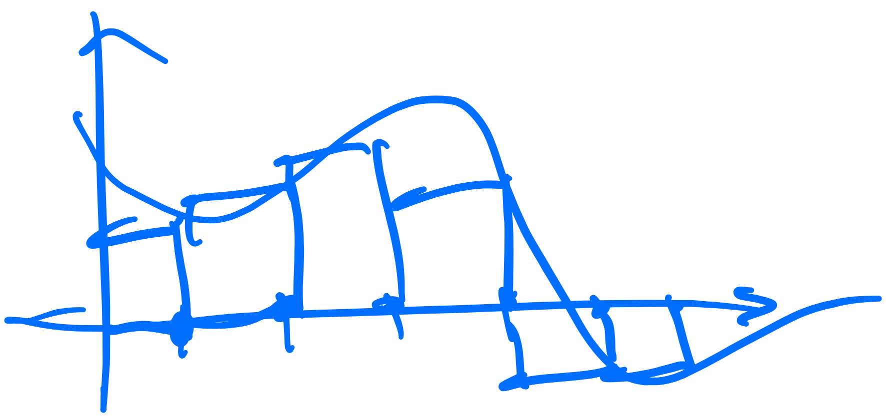
Practice problem 4

If we use a right endpoint approximation with four subintervals, then what is the approximation for $\int_0^4 f(x)\,dx$?



$$R_4 = 2 + 0 - 2 - 4$$

$$= \boxed{-4}$$



right end pts

Practice problem 5

Evaluate $\int_{-1}^{1}(x^2+2x+1)\,dx$.

symmetry

$\overset{\downarrow}{=} \int_{-1}^{1} x^2\,dx + \cancel{\int_{-1}^{1} 2x\,dx}^{\,0} + \int_{-1}^{1} dx$

(x is odd)

$= 2\int_0^1 x^2\,dx$ (x² is even) $\quad 2\int_0^1 dx$ (1 is even)

$= 2\frac{x^3}{3}\Big|_0^1 + 2x\Big|_0^1$

$= 2\cdot\left(\frac{1}{3}-0\right) + 2\cdot(1-0)$

$= \frac{2}{3}+2 = \boxed{\frac{8}{3}}$

standard

$= \frac{x^3}{3} + x^2 + x \Big|_{-1}^{1}$

$= \left(\frac{1}{3}+1+1\right) - \left(\frac{-1}{3}+1-1\right)$

$= \frac{1}{3}+2+\frac{1}{3}-1+1$

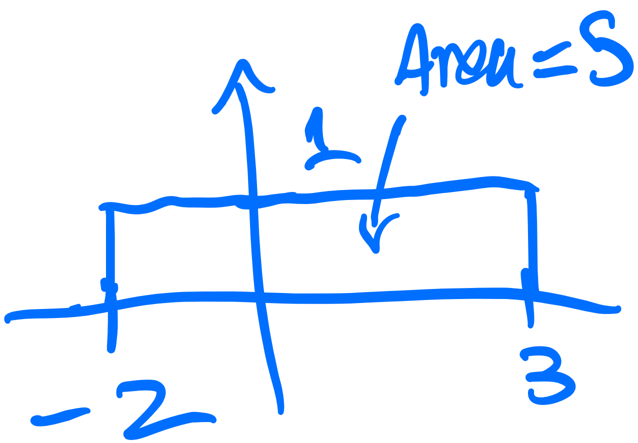
$= \frac{2}{3}+2+0 = 2+\frac{2}{3} = \frac{6}{3}+\frac{2}{3} = \boxed{\frac{8}{3}}$

Practice problem 6

If $\int_{-2}^{3} f(x)\,dx = 4$, then what is $\int_{-2}^{3} (f(x)+1)\,dx$?

$$= \int_{-2}^{3} f(x)\,dx + \int_{-2}^{3} dx$$

$$= 4 + \int_{-2}^{3} dx$$



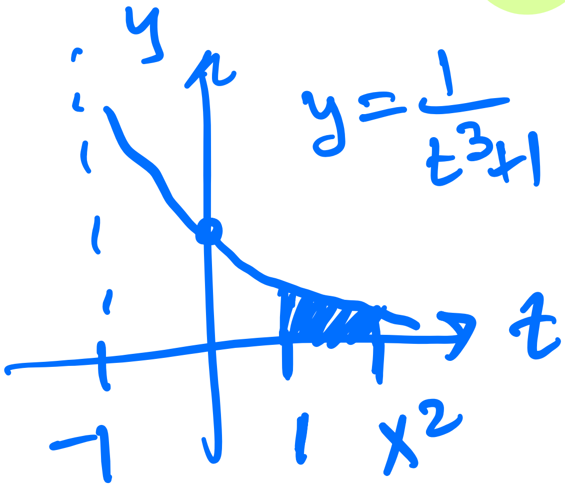
$$= 4 + 5$$

$$= \boxed{9}$$

WARNING: $\int f(x)g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx$

Practice problem 7

If $f(x) = \int_1^{x^2} \frac{1}{t^3+1}\, dt$, then what is $f'(x)$?



More basic question:

if $g(x) = \int_1^x \frac{1}{t^3+1}\, dt$, then

$$g'(x) = \frac{d}{dx}\int_1^x \frac{dt}{t^3+1} = \frac{1}{x^3+1}$$

FTC

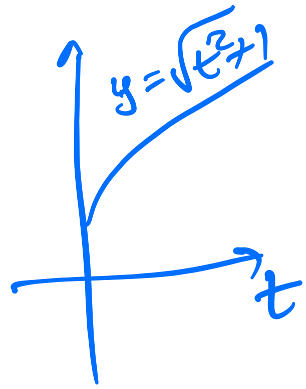
$$\frac{d}{dx}\left(\int_a^x h(t)\,dt\right) = h(x)$$

$$f(x) = g(x^2)$$

$$\rightarrow f'(x) = (g(x^2))' = g'(x^2)\cdot 2x$$

$$= \frac{1}{(x^2)^3+1}\cdot 2x = \boxed{\frac{2x}{x^6+1}}$$

Another such example:



$$f(x)=\int_0^{\sin x}\sqrt{t^2+1}\,dt$$

$\to f'(x) = ??$

Let $g(x) = \int_0^{x}\sqrt{t^2+1}\,dt$. Then $g'(x)=\sqrt{x^2+1}$

And $f(x) = g(\sin x)$, so

$$f'(x) = (g(\sin x))' = g'(\sin x)\cos x$$
$$= \sqrt{(\sin x)^2+1}\cdot\cos x$$

Practice problem 8

If $w'(t) = \dfrac{\ln t}{t}$ for $t \geq 1$ is the rate of growth of a child in pounds per year, find $\displaystyle\int_5^{10} w'(t)\,dt$ and give an interpretation of your answer.

$$\int_5^{10} w'(t)\,dt = \int_5^{10} \frac{\ln t}{t}\,dt \quad \begin{cases} \text{Set } u = \ln t \\ \text{so } du = \frac{1}{t}\,dt \end{cases}$$

Method 1: $\displaystyle\int \frac{\ln t}{t}\,dt = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln t)^2}{2} + C$

So $\displaystyle\int_5^{10} \frac{\ln t}{t}\,dt = \frac{(\ln t)^2}{2}\Big|_5^{10} = \frac{(\ln 10)^2}{2} - \frac{(\ln 5)^2}{2}$

Method 2: $\displaystyle\int_{t=5}^{t=10} \frac{\ln t}{t}\,dt = \int_{u=\ln 5}^{u=\ln 10} u\,du = \int_{\ln 5}^{\ln 10} u\,du = \frac{u^2}{2}\Big|_{\ln 5}^{\ln 10}$

$$= \frac{(\ln 10)^2}{2} - \frac{(\ln 5)^2}{2}$$

The meaning of

$$\int_5^{10} W'(t)\,dt = W(10) - W(5):$$

= change in weight at ages 5 and 10.

Practice problem 9a

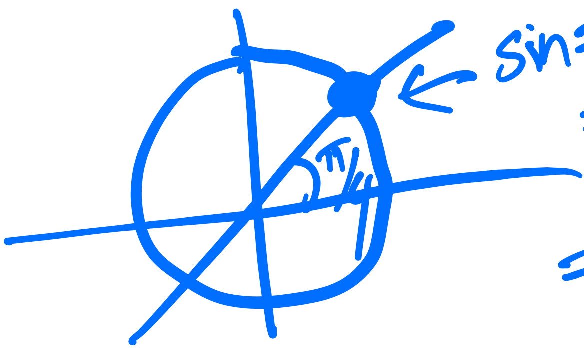
Evaluate $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2 x}{\cos^2 x}\, dx.$

$$= \int_0^{\pi/4} \left(\frac{1}{\cos^2 x} + 1 \right) dx$$

$$= \int_0^{\pi/4} \sec^2 x\, dx + \int_0^{\pi/4} dx$$

$$\sec x = \frac{1}{\cos x}$$

$$= \tan x \Big|_0^{\pi/4} + x \Big|_0^{\pi/4}$$



$$\sin = \cos = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$= \left(\tan \frac{\pi}{4} - \tan 0\right) + \left(\frac{\pi}{4} - 0\right)$$

$$= (1 - 0) + \frac{\pi}{4}$$

$$= \boxed{1 + \frac{\pi}{4}}$$

Practice problem 9b

Evaluate $\displaystyle\int_0^1 (x^{10} + 10^x)\,dx$.

$$\int_0^1 x^{10}\,dx + \int_0^1 10^x\,dx$$

$$= \frac{x^{11}}{11}\Big|_0^1 + \frac{10^x}{\ln 10}\Big|_0^1$$

Recall

$$(10^x)' = 10^x \ln 10$$

$$\left(\frac{10^x}{\ln 10}\right)' = 10^x$$

$$= \left(\frac{1}{11} - 0\right) + \left(\frac{10}{\ln 10} - \frac{10^0}{\ln 10}\right)$$

$$= \frac{1}{11} + \frac{10}{\ln 10} - \frac{1}{\ln 10} = \boxed{\frac{1}{11} + \frac{9}{\ln 10}}$$

Practice problem 9c

Evaluate $\int \left(\frac{1+r}{r}\right)^2 dr$.

$$\left(\frac{1+r}{r}\right)^2 = \frac{(1+r)^2}{r^2} = \frac{1+2r+r^2}{r^2} = \frac{1}{r^2}+\frac{2}{r}+1$$

$$\Rightarrow \int \left(\frac{1+r}{r}\right)^2 dr = \int \left(\frac{1}{r^2}+\frac{2}{r}+1\right) dr$$

$$= \int \frac{1}{r^2}dr + 2\int \frac{1}{r}dr + \int dr$$

$$= \int r^{-2}dr + 2\ln r + r + C$$

$$= \frac{r^{-1}}{-1} + 2\ln r + r + C = \boxed{\frac{-1}{r} + 2\ln r + r + C}$$

$$\int \frac{1}{r^2}dr = -\frac{1}{r} + C$$

Practice problem 9d

Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

$$\sqrt{x}' = \frac{1}{2\sqrt{x}} \qquad \left(e^{g(x)}\right)' = e^{g(x)} \cdot g'(x)$$

Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}dx$. Then $\frac{dx}{\sqrt{x}} = 2du$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int e^u \cdot 2du = 2\int e^u\,du$$

$$= 2e^u + C$$

$$= \boxed{2e^{\sqrt{x}} + C}$$

Check it works...

Practice problem 9e

Evaluate $\displaystyle\int_5^{10} \frac{dt}{(t-4)^2}$.

$t=5 \Rightarrow u=1$

$t=10 \Rightarrow u=6$

Let $u=t-4$, so $du=dt$

$$\int_5^{10} \frac{dt}{(t-4)^2} = \int_1^6 \frac{1}{u^2}\,du = \int_1^6 u^{-2}\,du$$

$$= \frac{u^{-2+1}}{-2+1}\Big|_1^6 = \frac{u^{-1}}{-1}\Big|_1^6$$

$$= \frac{-1}{u}\Big|_1^6$$

$$= -\frac{1}{6} - \left(-\frac{1}{1}\right)$$

$$= -\frac{1}{6} + 1 = 1 - \frac{1}{6} = \boxed{\frac{5}{6}}$$

Practice problem 9f

Evaluate $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$.

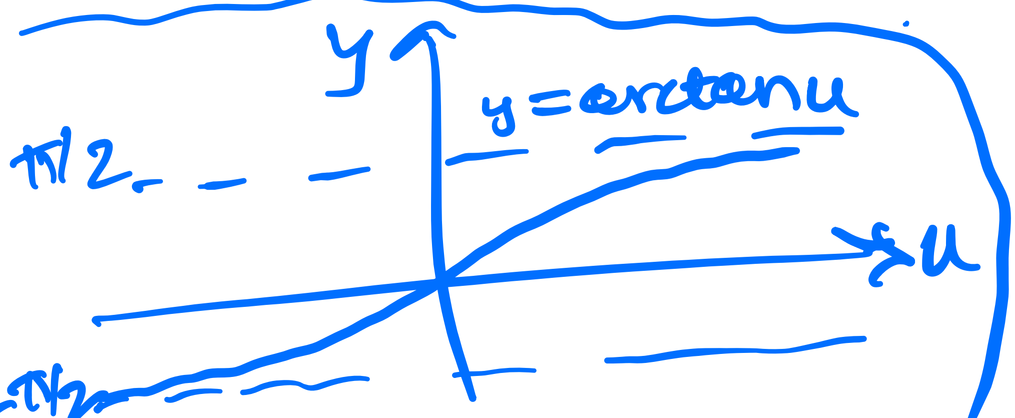
$e^{2x} = (e^x)^2$

Let $u = e^x$, so $du = e^x\,dx$

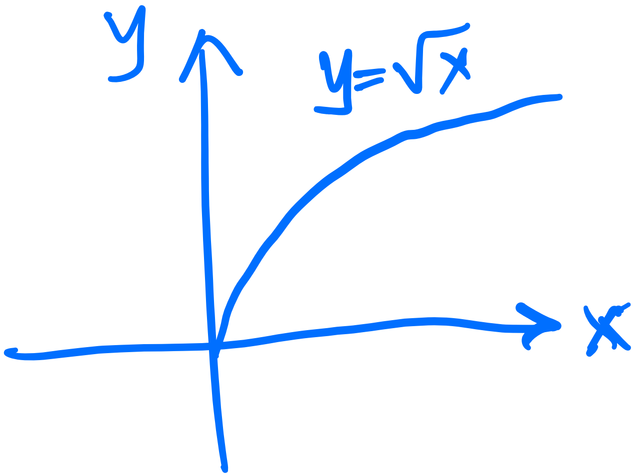
and $1 + e^{2x} = 1 + (e^x)^2 = 1 + u^2$, so

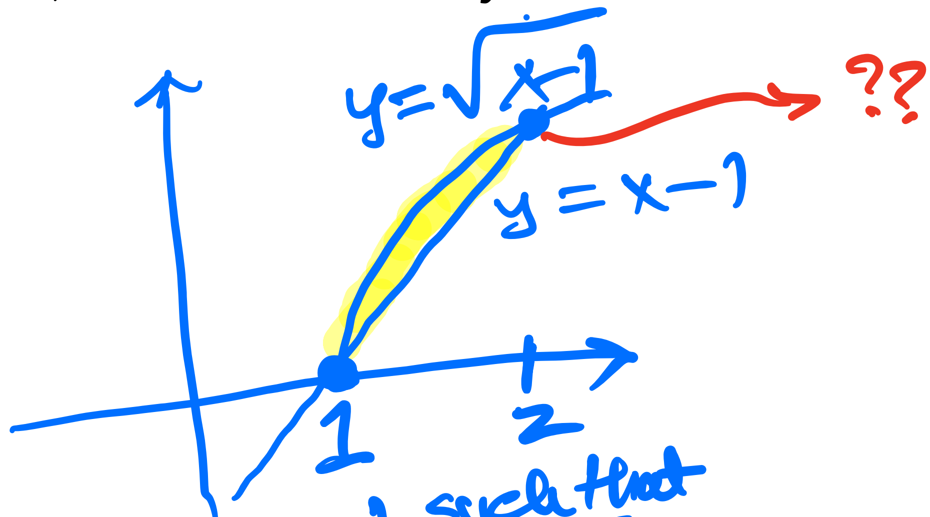
$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \arctan(u) + C$$

$$= \boxed{\arctan(e^x) + C}$$



Sketch the region bounded by $y = \sqrt{x-1}$ and $x - y = 1$ and determine its area.





$x - y = 1 \iff x = 1 + y \iff y = x - 1$

Want $x > 1$ such that

$x - 1 = \sqrt{x-1}$

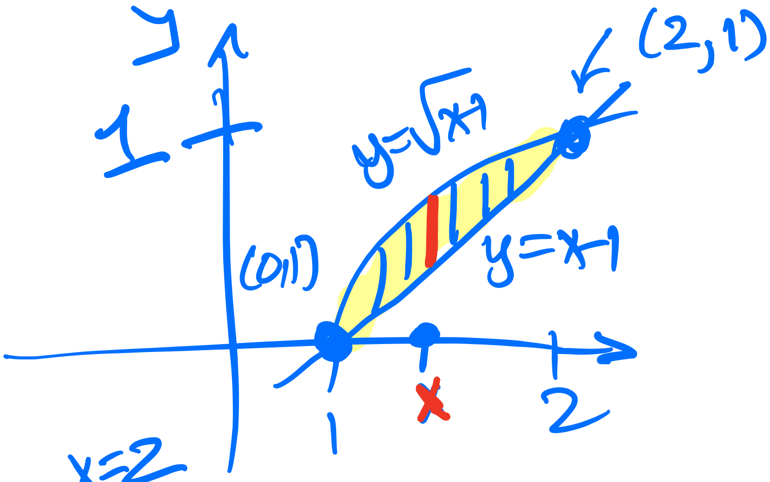
$\Rightarrow (x-1)^2 = \sqrt{x-1}^{\,2}$

$(x-1)^2 = x - 1$

$x^2 - 2x + 1 = x - 1$ OR $x - 1 = 1 \Rightarrow x = 2$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0 \Rightarrow x = 1$ or 2



$1 \le x \le 2$

or

$0 \le y \le 1$

$$\text{Area} = \int_{x=1}^{x=2} \left(\sqrt{x-1} - (x-1)\right) dx$$

Let $u = x-1$, $du = dx$

$x=1 \rightarrow u=1-1=0$

$x=2 \rightarrow u=2-1=1$

$$= \int_0^1 (\sqrt{u} - u)\, du$$

$$= \int_0^1 (u^{1/2} - u)\, du = \left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{u^2}{2}\right)\Bigg|_0^1$$

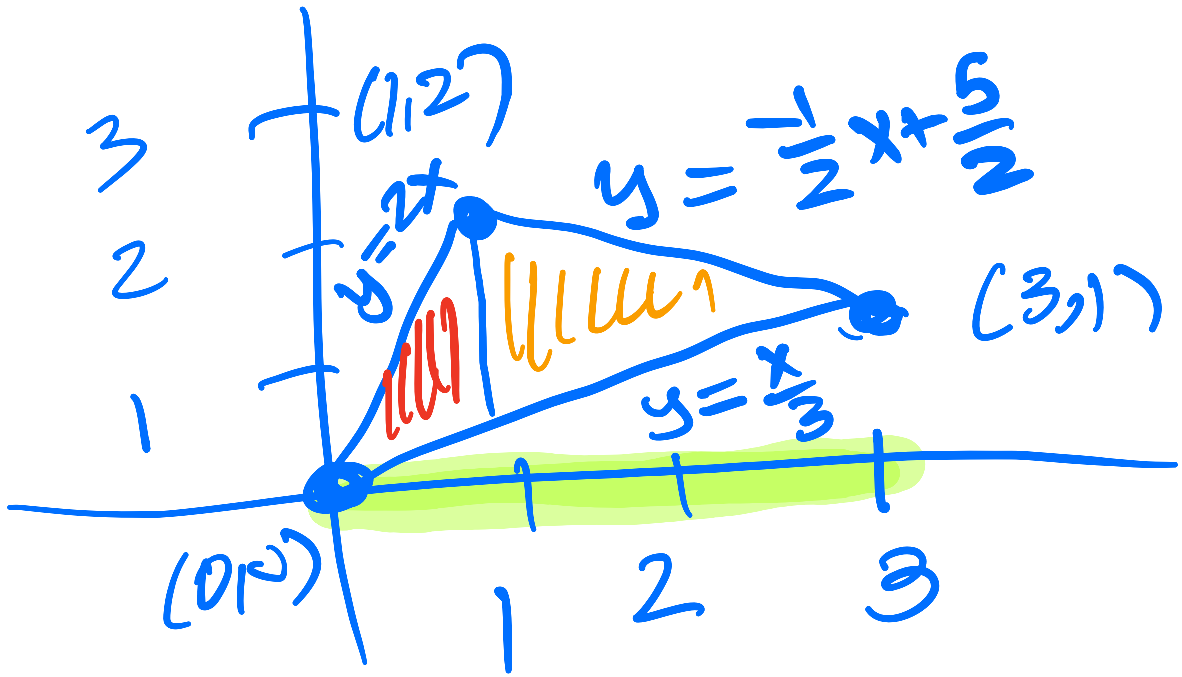
$$= \left(\frac{u^{3/2}}{3/2} - \frac{u^2}{2}\right)\Bigg|_0^1$$

$$= \left(\frac{1}{3/2} - \frac{1}{2}\right) - (0-0)$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \boxed{\frac{1}{6}}$$

Practice problem 11

Use calculus to find the area of the triangle with vertices $(0, 0)$, $(3, 1)$, and $(1, 2)$.

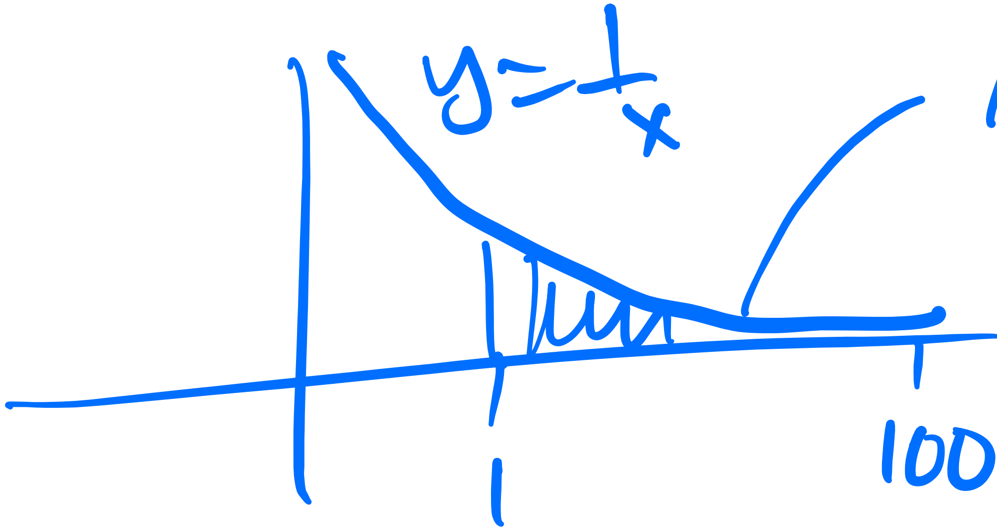


$$A=\int_0^1\left(2x-\frac{x}{3}\right)dx+\int_1^3\left(-\frac{1}{2}x+\frac{5}{2}-\frac{x}{3}\right)dx$$

$$=[\text{PAIN}]=5/2$$

Practice problem 12a

What is the area of the region under the curve $y = 1/x$ for $1 \leq x \leq 100$?



$$\text{Area} = \int_1^{100} \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^{100}$$

$$= \ln 100 - \cancelto{0}{\ln 1}$$

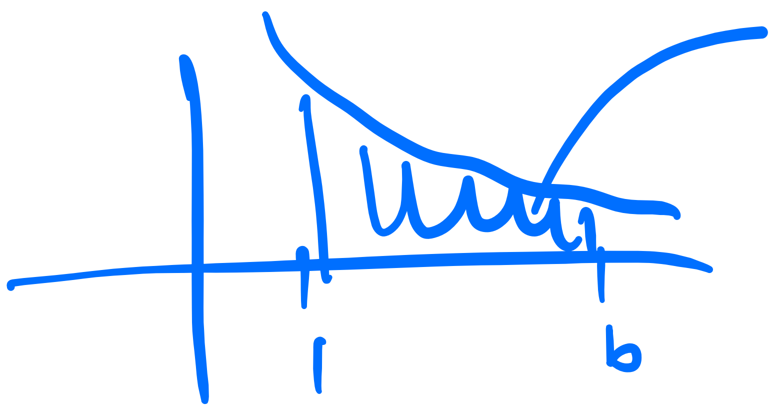
$$= \ln 100$$

For all $a > 0$,

$$\boxed{\int_1^a \frac{dx}{x} = \ln a}$$

Practice problem 12b

What happens to the area in part (a) as the right endpoint tends to ∞?



$$\text{Area} = \int_1^b \frac{1}{x}\,dx$$

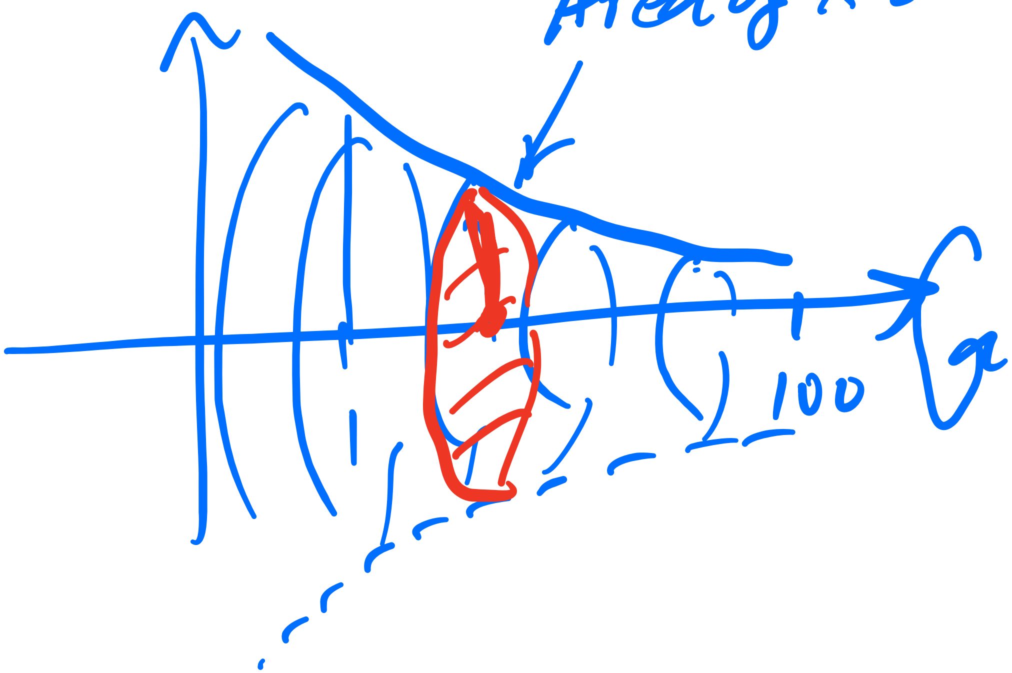
$$= \ln b - \ln 1 = \ln b$$

As $b \to \infty$, $\ln(b) \longrightarrow \infty$

Practice problem 12c

What is the volume of the solid obtained by revolving $y = 1/x$ for $1 \le x \le 100$ around the x-axis?

Area of x-slice $= \pi \cdot \left(\frac{1}{x}\right)^2 = \frac{\pi}{x^2}$



$$V = \int_1^{100} A(x)\,dx$$

$$= \int_1^{100} \frac{\pi}{x^2}\,dx$$

$$= \pi \int_1^{100} \frac{1}{x^2}\,dx$$

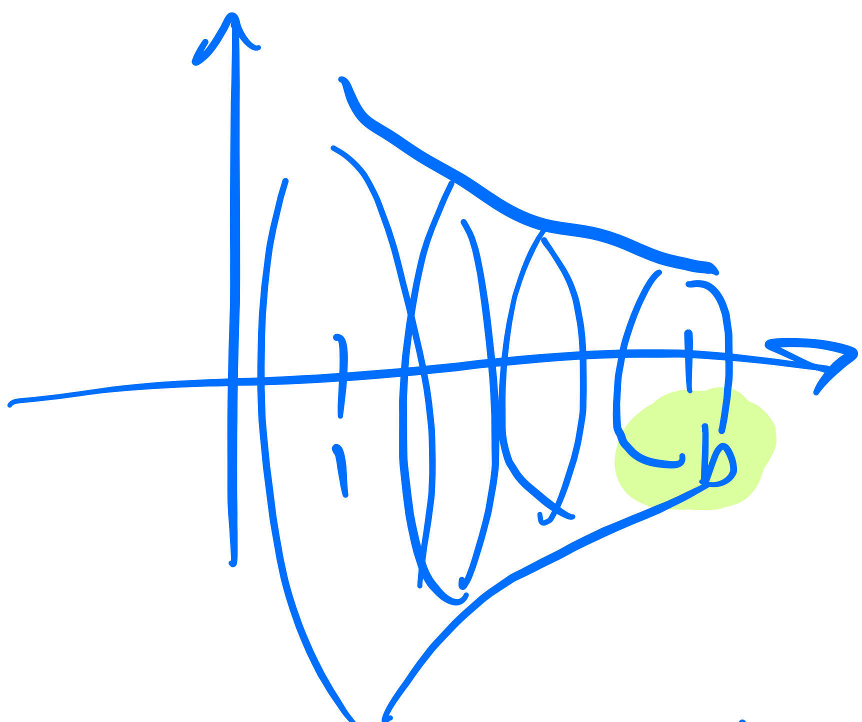
$$= \pi \int_1^{100} x^{-2}\,dx$$

$$= \pi \left(\frac{x^{-1}}{-1}\right)\Big|_1^{100} = \pi\left(-\frac{1}{x}\right)\Big|_1^{100} = \pi\left(-\frac{1}{100} - (-1)\right)$$

$$= \pi\left(-\frac{1}{100} + 1\right) = \frac{99}{100}\pi$$

Practice problem 12d

What happens to the volume in part (c) as the right endpoint tends to ∞?



$$V = \int_1^b A(x)\,dx$$

$$= \int_1^b \frac{\pi}{x^2}\,dx$$

$$= \pi \int_1^b \frac{1}{x^2}\,dx$$

$$= \pi\left(-\frac{1}{x}\Big|_1^b\right) = \pi\left(-\frac{1}{b} - (-1)\right)$$

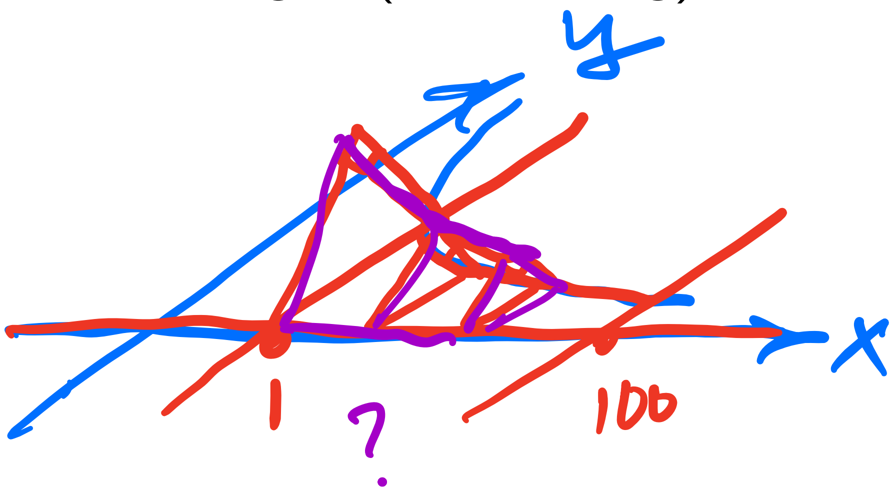
$$= \pi\left(1 - \frac{1}{b}\right) \longrightarrow \pi(1-0) = \pi$$

as $b \to \infty$

See Torricelli's paradox

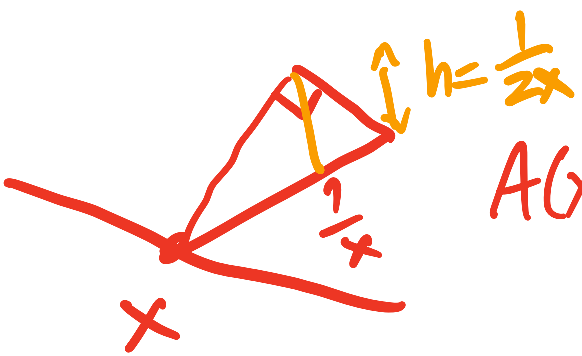
Practice problem 12e

What is an integral for the volume of the solid whose base is the region bounded by $y = 1/x$, $y = 0$, $x = 1$, and $x = 100$ with cross-sections perpendicular to the x-axis being right triangles whose height (shorter leg) is half their base (longer leg)?



$$V = \int_1^{100} A(x)\,dx$$

$$= \int_1^{100} \frac{1}{4x^2}\,dx$$



$$A(x) = \frac{1}{2} bh$$

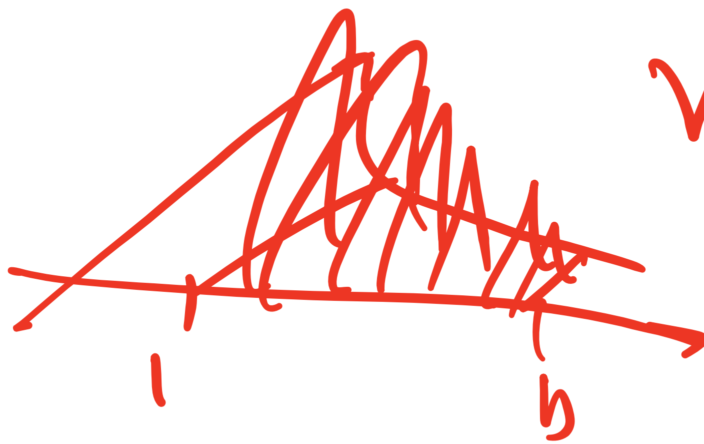
$$= \frac{1}{2}\frac{1}{x}\frac{1}{2x} = \frac{1}{4x^2}$$

Practice problem 12f

What happens to the volume in part (e) as the right endpoint tends to ∞?

Answer 1

Answer 2:



$$V = \int_1^b \frac{1}{4x^2}\,dx$$

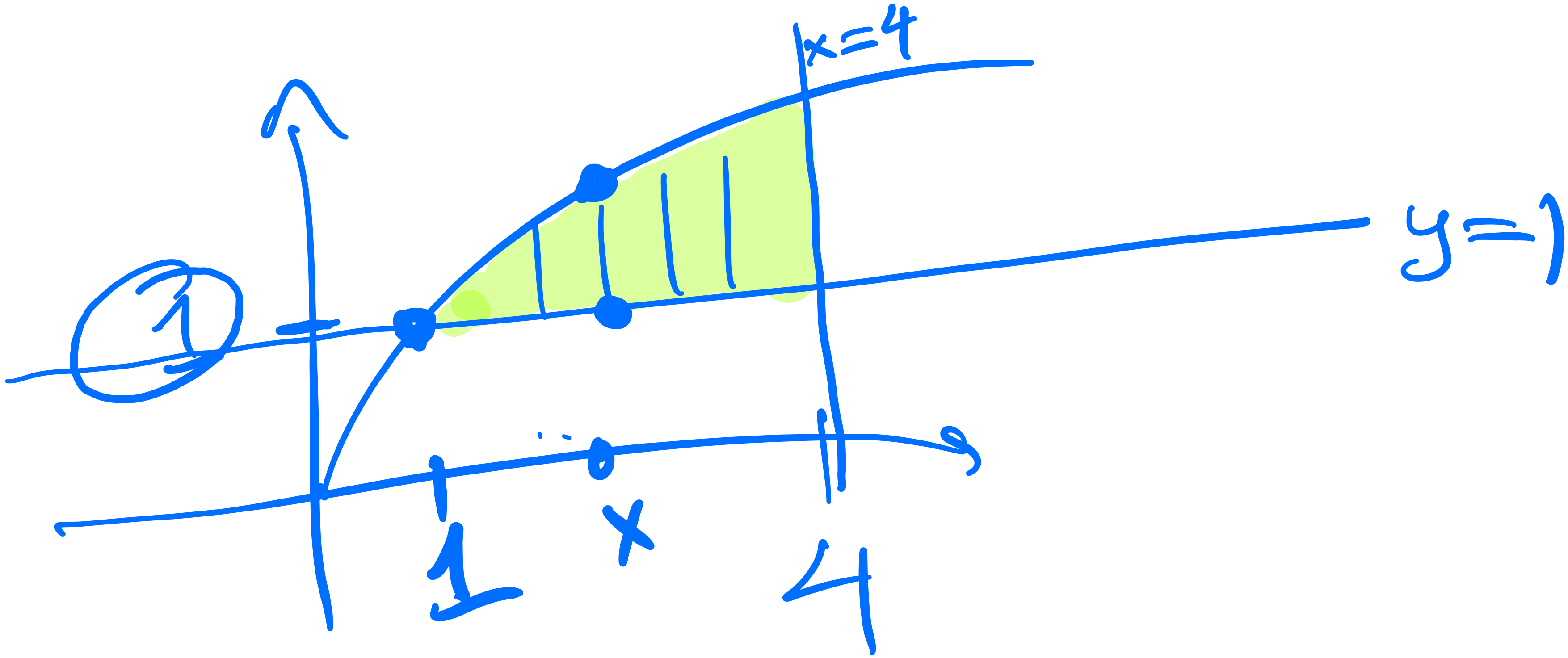
$$= \frac{1}{4}\int_1^b \frac{1}{x^2}\,dx$$

$$= \frac{1}{4}\left(-\frac{1}{x}\right)\Big|_1^b = \frac{1}{4}\left(1-\frac{1}{b}\right)$$

$$\longrightarrow \frac{1}{4} \text{ as } b\to\infty$$

Practice problem 13a

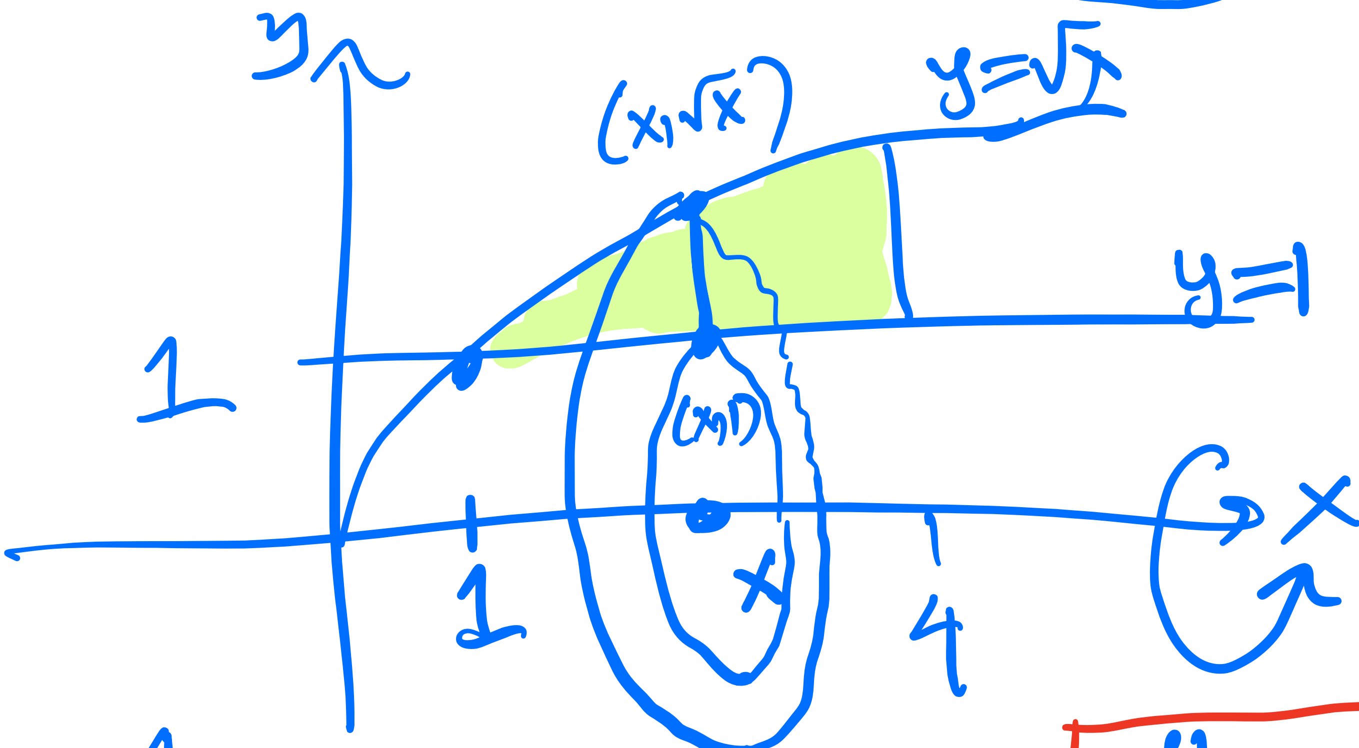
Consider the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$. Set up, but do not evaluate, an integral for the area of this region.



$$\boxed{A = \int_1^4 (\sqrt{x} - 1)\,dx}$$

Practice problem 13b

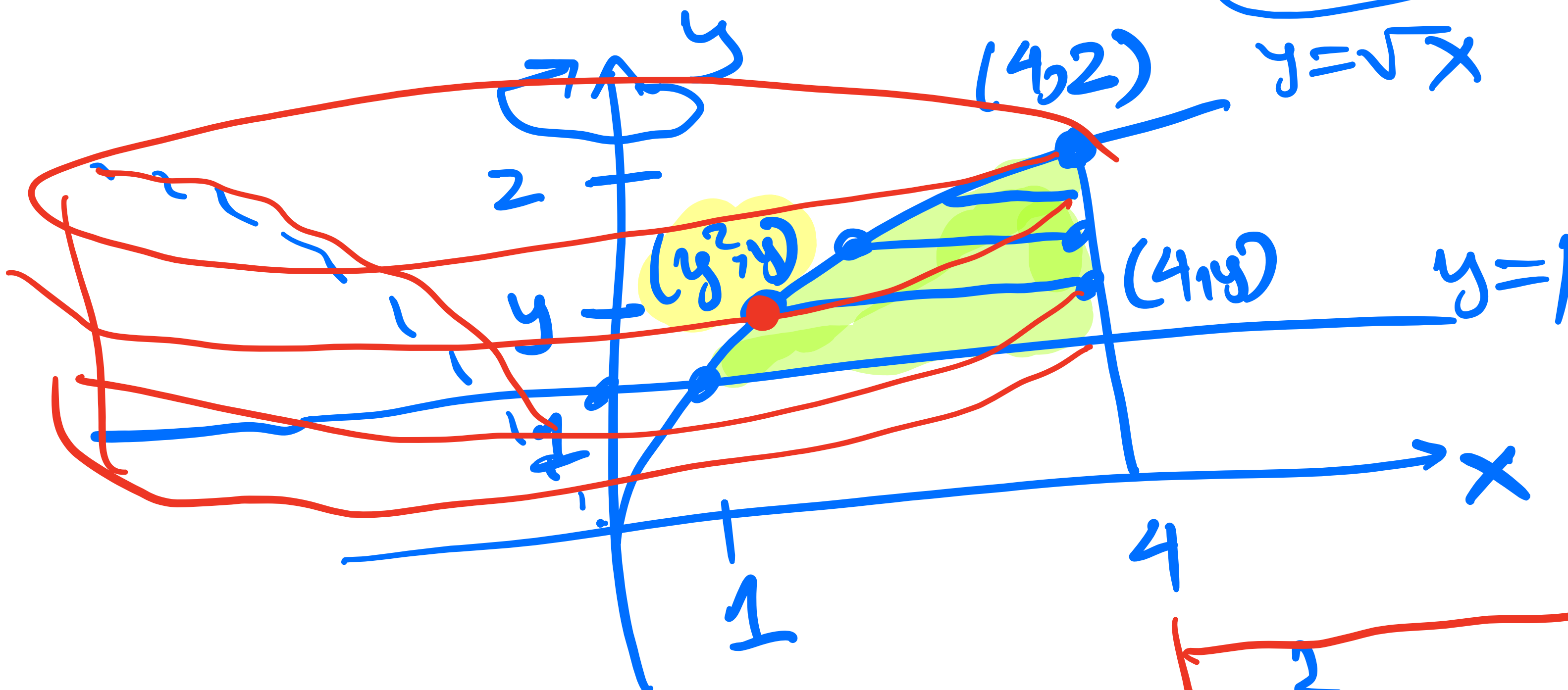
Consider the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region around the x-axis.



$$V = \int_1^4 \pi \Big(\underbrace{\sqrt{x}^2}_{(\text{outer rad.})^2} - \underbrace{1^2}_{(\text{inner rad.})^2} \Big)\,dx = \boxed{\int_1^4 \pi (x-1)\,dx}$$

Practice problem 13c

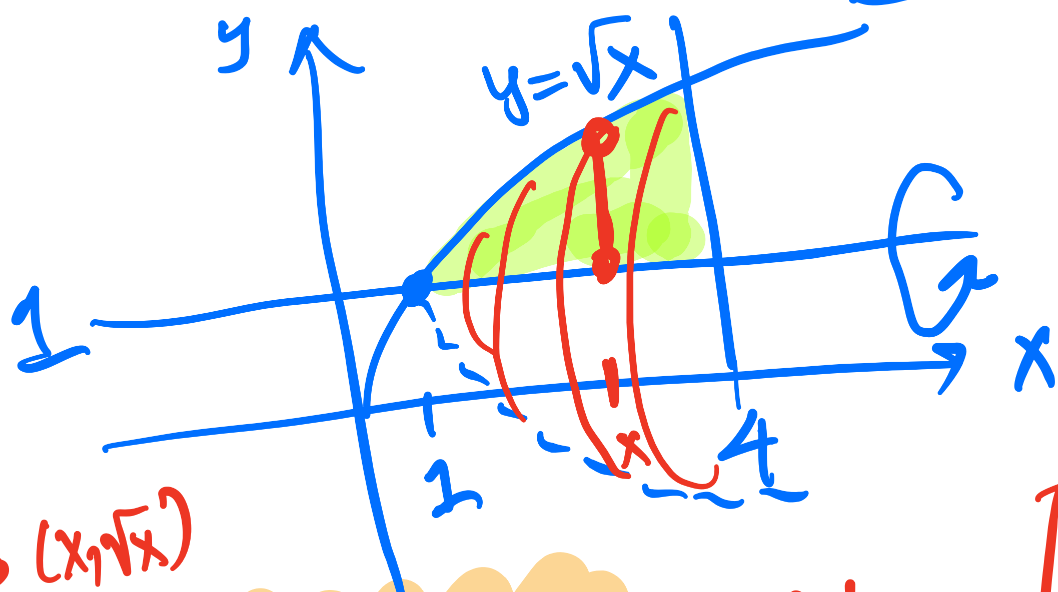
Consider the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region around the y-axis.



$y=\sqrt{x}$
$\Rightarrow x=y^2$

$$V=\int_1^2 \pi\left(\underbrace{4^2}_{(\text{outer rad.})^2}-\underbrace{(y^2)^2}_{(\text{inner rad.})^2}\right)dy=\boxed{\int_1^2 \pi\left(16-y^4\right)dy}$$

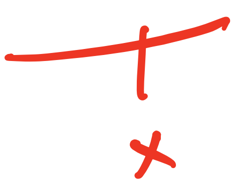
Consider the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region around the line $y = 1$.



$(x, \sqrt{x})$

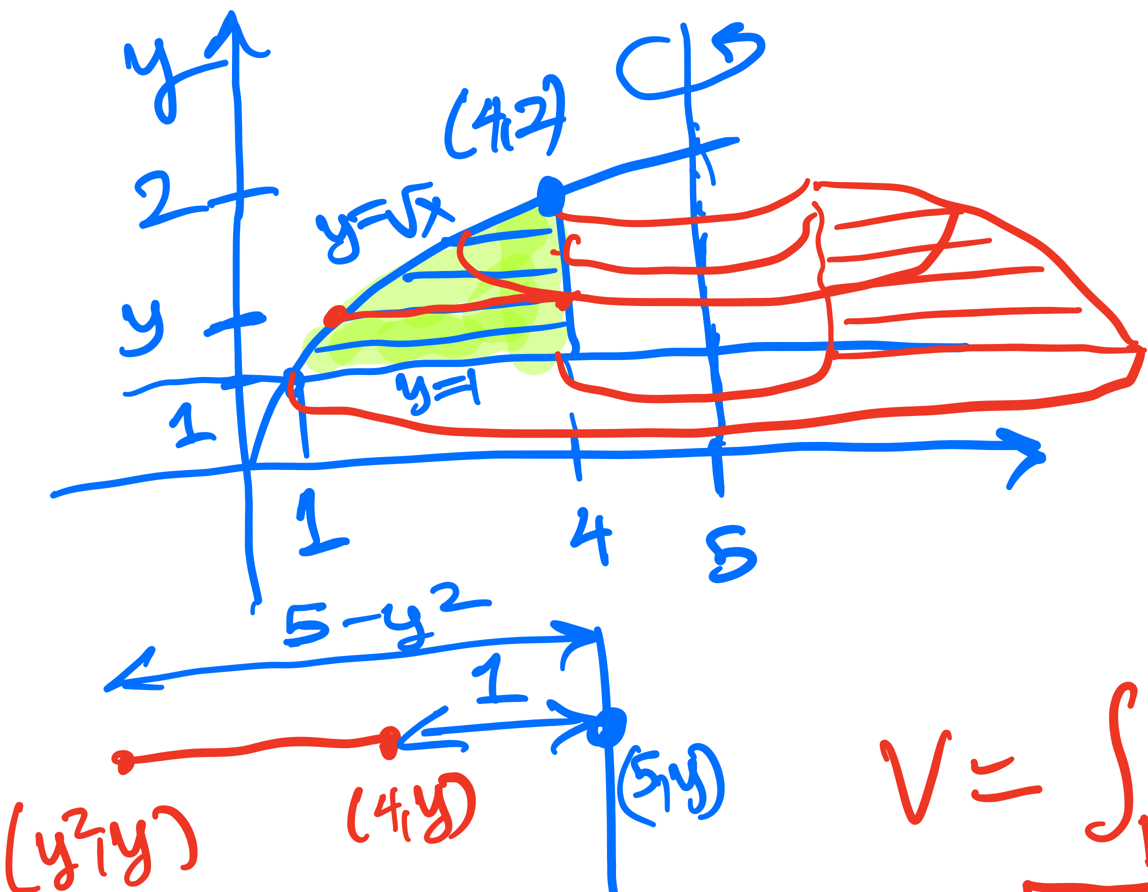
$(x, 1)$

length $= \sqrt{x} - 1 \Rightarrow V =$ $\boxed{\int_1^4 \pi (\sqrt{x} - 1)^2 \, dx}$



Practice problem 13e

Consider the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region around the line $x = 5$.



$$V = \int_1^2 \pi\left(\begin{pmatrix}\text{outer}\\ \text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{radius}\end{pmatrix}^2\right) dy$$

$$= \boxed{\int_1^2 \pi\left((5-y^2)^2 - 1\right) dy}$$