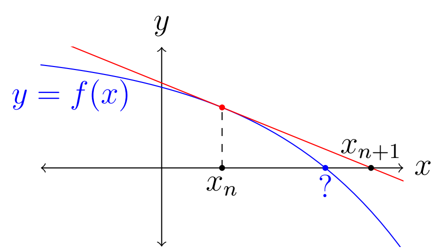
Applications: Newton's Method

Newton's method is a technique for solving an equation of the form $f(x) = 0$ not by algebra, but by *iteration*: it finds a solution as the limit of an iterative process. Starting with a number x_1, we set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

as long as this makes sense (that is, as long as $f'(x_n)$ is nonzero). What this procedure does is illustrated below: draw the tangent line to the graph of $y = f(x)$ at the point where $x = x_n$ and where this line meets the x-axis is x_{n+1}.



When the numbers $x_1, x_2, x_3, \ldots$ converge to a solution of $f(x) = 0$, they often do so *quickly*: the number of correct digits in x_n tends to double after each iteration.

Example. Let $f(x) = x^2 - 5$. The recursion in Newton's method here is

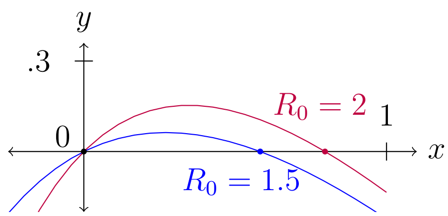
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{x_n^2 + 5}{2x_n}.$$

The table below illustrates two iterations, where $x_1 = 1$ and $x_1 = 2$. Both sequences of numbers x_n converge to $\sqrt{5}$, and by the 6th iterate x_n matches $\sqrt{5}$ in 8 digits.[1]

n	x_n	x_n
1	1.0000000	2.0000000
2	3.0000000	2.2500000
3	2.3333333	2.2361111
4	2.2380952	2.2360679
5	2.2360688	2.2360679
6	2.2360679	2.2360679

[1] The truth is even better: the sequence starting with $x_1 = 1$ has x_6 matching $\sqrt{5}$ in 13 digits and the sequence starting with $x_1 = 2$ has x_6 matching $\sqrt{5}$ in over 30 digits.

Example. Graphs of $y = 1 - x - e^{-R_0 x}$ for $R_0 = 1.5$ and $R_0 = 2$ are below.



The recursion for solving $f(x) = 0$ where $f(x) = 1 - x - e^{-R_0 x}$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1 - x_n - e^{-R_0 x_n}}{-1 + R_0 e^{-R_0 x_n}} = \frac{(R_0 x_n + 1)e^{-R_0 x_n} - 1}{R_0 e^{-R_0 x_n} - 1}.$$

The table below gives iterations where $x_1 = 1$ for $R_0 = 1.5$ and $R_0 = 2$.

n	x_n ($R_0 = 1.5$)	x_n ($R_0 = 2$)
1	1.00000000	1.00000000
2	0.66461962	0.81443874
3	0.58929685	0.79701507
4	0.58286321	0.79681215
5	0.58281164	0.79681213
6	0.58281164	0.79681213

This is a real-world example from epidemilogy: the solution to $1 - x - e^{-R_0 x} = 0$, where R_0 is the basic reproduction number for an epidemic and $R_0 > 1$, is the prediction for the proportion of the population to get infected (about 58% if $R_0 = 1.5$ and 79% if $R_0 = 2$). For more, see here (watch 15:50-20:00) and here.

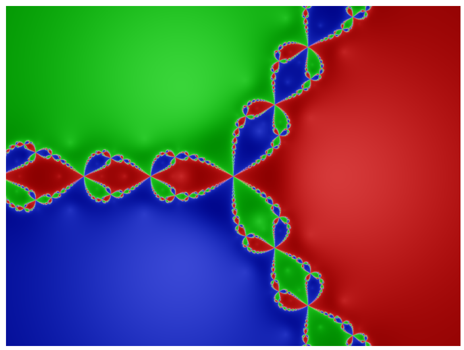
For some functions $f(x)$ and starting points x_1, Newton's method does not converge. There are conditions under which Newton's method is guaranteed to converge: see here, although the setting there is a multivariable form of Newton's method. In this introductory calculus course we don't discuss such conditions.

Here are a few examples where Newton's method is used.

Application 1. Solving Kepler's equation $a + b\sin x = x$ for x, where a and b are constants, is used for GPS calculations (see slide 27 here).

Application 2. Computing reciprocal square roots for video games. This is needed to rescale vectors to have length 1, and must be done thousands of times per second for lighting and shading effects. To find $x = 1/\sqrt{a}$ where a is known, rewrite it as $1/x^2 - a = 0$. Newton's method for this is $x_{n+1} = \frac{3}{2}x_n - \frac{a}{2}x_n^3$, which lets us get good estimates on $1/\sqrt{a}$ without needing square roots or division (except by 2, which is a simple bit shift in computer code).

Application 3. For a polynomial $f(x)$ with more than two roots in the complex numbers, color complex numbers based on which root of $f(x)$ they tend to by Newton's method. This leads to fractal images called the Julia sets of $x - f(x)/f'(x)$. Below is the case of $f(x) = x^3 - 1$. Red points under Newton's method tend to the root 1, green points tend to the root $-1/2 + (\sqrt{3}/2)i$ and blue points tend to the root $-1/2 - (\sqrt{3}/2)i$. A video about such fractals by Grant Sanderson is here.



Equations arising in applications often involve many variables, and there may be no way to see a solution to such equations in a graph. There is a version of Newton's method for solving systems of nonlinear equations in several variables (based on multivariable calculus and linear algebra), and when it works it tends to do so quickly, just like in one variable. Here are applications of the multivariable Newton's method.

Application 4. Inverse kinematics problems (robotics, video game animation).

Application 5. Multivariable optimization problems are solved with a higher-dimensional analogue of $f'(x) = 0$, which is a type of equation to which Newton's method can be used. Examples of this occur in logistic regression and the XGBoost algorithm, which are both very widely used in machine learning.

Application 6. The numerical solution of nonlinear differential equations. Discretization turns differential equations into algebraic equations (polynomials), and Newton's method is the most widely used numerical technique for solving nonlinear algebraic equations. See here.

Most equations do not have direct formulas for their solutions. This is why it's important to appreciate iterative solution methods, of which there are many "named" types. We have seen Newton's method. Others include gradient descent for finding the minimum of a function and k-means clustering in data mining.