CHAIN RULE PROBLEMS

The chain rule says $(f(g(x)))' = f'(g(x))g'(x)$, or $(f(u))' = f'(u)u'(x)$ if $u = g(x)$. To carry out the chain rule, know basic derivatives well so you can build on that. In the table below, observe how each basic derivative in the first column generalizes by the chain rule with $g(x)$ in place of x in the third column. (**Note**: The derivatives of $\ln x$ and $\arctan x$ below are presented in Sections 3.6 and 3.5, after Section 3.4 where the chain rule is first introduced.)

Basic derivative	**Composition**	**Chain rule example**
$(x^c)' = cx^{c-1}$ for constant c	$g(x)^c = u^c$ for $u = g(x)$ and constant c	$(g(x)^c)' = cg(x)^{c-1}g'(x)$ for constant c
$(x^3)' = 3x^2$	$g(x)^3 = u^3$ for $u = g(x)$	$((g(x))^3)' = 3g(x)^2g'(x)$
$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$\sqrt{g(x)} = \sqrt{u}$ for $u = g(x)$	$(\sqrt{g(x)})' = \frac{1}{2\sqrt{g(x)}}g'(x)$
$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$	$\frac{1}{g(x)} = \frac{1}{u}$ for $u = g(x)$	$\left(\frac{1}{g(x)}\right)' = \frac{-1}{g(x)^2}g'(x)$
$(e^x)' = e^x$	$e^{g(x)} = e^u$ for $u = g(x)$	$(e^{g(x)})' = e^{g(x)}g'(x)$
$(\ln x)' = \frac{1}{x}$	$\ln g(x) = \ln u$ for $u = g(x)$	$(\ln g(x))' = \frac{1}{g(x)}g'(x)$
$(\sin x)' = \cos x$	$\sin(g(x)) = \sin u$ for $u = g(x)$	$(\sin(g(x)))' = \cos(g(x))g'(x)$
$(\cos x)' = -\sin x$	$\cos(g(x)) = \cos u$ for $u = g(x)$	$(\cos(g(x)))' = -\sin(g(x))g'(x)$
$(\tan x)' = \sec^2 x$	$\tan(g(x)) = \tan u$ for $u = g(x)$	$(\tan(g(x)))' = \sec^2(g(x))g'(x)$
$(\arctan x)' = \frac{1}{x^2+1}$	$\arctan(g(x)) = \arctan u$ for $u = g(x)$	$(\arctan(g(x)))' = \frac{1}{g(x)^2+1}g'(x)$

Problems.

Work out derivatives of all of the following functions. Wait to do the last five until after you have learned the derivatives of $\ln x$ and $\arctan x$. Answers are on the next page, which you can use to check if you did each problem correctly.

(1) $\sin^2 x$ (This *means* $(\sin x)^2$, so the "outer" function is x^2, *not* $\sin x$.)
(2) $\sin(\sin x)$
(3) $\sin(\sin(\sin x))$
(4) $\sin(\sin(\sin(\sin x)))$
(5) $\sqrt{x^2 + 5x}$
(6) $(x^3 + x^2 + 1)^4$
(7) $e^{-x^2/5}$
(8) $(\sqrt{x} + x^3)^8$
(9) $\dfrac{1}{\cos x}$
(10) $\cos\left(\dfrac{1}{x}\right)$
(11) $3^{\cos x}$
(12) $\cos(5x)$
(13) $3^{\cos(5x)}$
(14) $\cos(3^x)$
(15) $\cos(\sqrt{x})$
(16) $\sqrt{\cos x}$
(17) $\tan(e^{-x} + 4x^2)$.
(18) $\tan((2x + 1)^3 + x)$
(19) $\tan^3 x$ (This *means* $(\tan x)^3$, so the "outer" function is x^3, *not* $\tan x$.)
(20) $\sin(x^3 - x)$
(21) $((x^2 - x)^3 + x + 2)^4$
(22) $\dfrac{1}{x^3 + 3^x}$.
(23) $e^{\sin(x^2)}$
(24) $\sin(e^{x^2})$
(25) $\sin^2(e^x)$. (This *means* $(\sin(e^x))^2$, so the "outer" function is x^2, *not* $\sin x$.)
(26) $\ln(4x + 3)$
(27) $\dfrac{1}{(\ln x)^4}$
(28) $\cos(\ln(x^3 + 1))$
(29) $(\ln(\cos x))^3 + 1$
(30) $\arctan(2\sin(x^3))$

Answers.

(1) $2(\sin x)(\cos x)$. (Let $u = \sin x$ to make the function u^2.)

(2) $(\cos(\sin x))\cos x$. (Let $u = \sin x$ to make the function $\sin(u)$.)

(3) $(\cos(\sin(\sin x)))\cos(\sin x)\cos x$. (Let $u = \sin(\sin x)$ to make the function $\sin(u)$.)

(4) $(\cos(\sin(\sin(\sin x))))(\cos(\sin(\sin x)))\cos(\sin x)\cos x$. (Let $u = \sin(\sin(\sin x))$ to make the function $\sin(u)$.)

(5) $\dfrac{1}{2\sqrt{x^2+5x}}(2x+5)$. (Let $u = x^2+5x$ to make the function $\sqrt{u} = u^{1/2}$.)

(6) $4(x^3+x^2+1)^3(3x^2+2x)$. (Let $u = x^3+x^2+1$ to make the function u^4.)

(7) $e^{-x^2/5}(-2x/5)$. (Let $u = -x^2/5$ to make the function e^u.)

(8) $8(\sqrt{x}+x^3)^7\left(\dfrac{1}{2\sqrt{x}}+3x^2\right)$. (Let $u = \sqrt{x}+x^3$ to make the function u^8.)

(9) $\dfrac{-1}{(\cos x)^2}(-\sin x) = \dfrac{\sin x}{\cos^2 x}$. (Let $u = \cos x$ to make the function $1/u$.)

(10) $-\sin\left(\dfrac{1}{x}\right)\left(\dfrac{-1}{x^2}\right) = \dfrac{\sin(1/x)}{x^2}$. (Let $u = 1/x$ to make the function $\cos u$.)

(11) $3^{\cos x}(\ln 3)(-\sin x) = -(\ln 3)3^{\cos x}(\sin x)$. (Let $u = \cos x$ to make the function 3^u.)

(12) $-\sin(5x)5 = -5\sin(5x)$. (Let $u = 5x$ to make the function $\cos u$.)

(13) $3^{\cos(5x)}(\ln 3)(-\sin(5x))5 = -5(\ln 3)3^{\cos(5x)}\sin(5x)$. (Let $u = \cos(5x)$ to make the function 3^u.)

(14) $-\sin(3^x)3^x(\ln 3) = -(\ln 3)(\sin(3^x))3^x$. (Let $u = 3^x$ to make the function $\cos u$.)

(15) $-\sin(\sqrt{x})\dfrac{1}{2\sqrt{x}} = \dfrac{-\sin(\sqrt{x})}{2\sqrt{x}}$. (Let $u = \sqrt{x}$ to make the function $\cos u$.)

(16) $\dfrac{1}{2\sqrt{\cos x}}(-\sin x) = \dfrac{-\sin x}{2\sqrt{\cos x}}$. (Let $u = \cos x$ to make the function $\sqrt{u}$.)

(17) $(\sec^2(e^{-x}+4x^2))(-e^{-x}+8x)$. (Let $u = e^{-x}+4x^2$ to make the function $\tan u$.)

(18) $(\sec^2((2x+1)^3+x))(3(2x+1)^2+1)(2) = 2(3(2x+1)^2+1)\sec^2((2x+1)^3+x)$. (Let $u = (2x+1)^3+x$ to make the function $\tan u$.)

(19) $3\tan^2 x\sec^2 x$. (Let $u = \tan x$ to make the function u^3.)

(20) $(\cos(x^3-x))(3x^2-1)$. (Let $u = x^3-x$ to make the function $\sin u$.)

(21) $4((x^2-x)^3+x+2)^3(3(x^2-x)^2(2x-1)+1)$. (Let $u = (x^2-x)^3+x+2$ to make the function u^4.)

(22) $\dfrac{-1}{(x^3+3^x)^2}(3x^2+3^x\ln 3)$. (Let $u = x^3+3^x$ to make the function $1/u$.)

(23) $e^{\sin(x^2)}(\cos(x^2))(2x) = 2x\cos(x^2)e^{\sin(x^2)}$. (Let $u = \sin(x^2)$ to make the function e^u.)

(24) $(\cos(e^{x^2}))e^{x^2}(2x) = 2xe^{x^2}\cos(e^{x^2})$. (Let $u = e^{x^2}$ to make the function $\sin u$.)

(25) $2(\sin(e^x))(\cos(e^x))e^x = 2e^x\sin(e^x)\cos(e^x)$. (Let $u = \sin(e^x)$ to make the function u^2.)

(26) $\dfrac{1}{4x+3}(4) = \dfrac{4}{4x+3}$. (Let $u = 4x+3$ to make the function $\ln u$.)

(27) $-\dfrac{4}{(\ln x)^5}\dfrac{1}{x} = \dfrac{-4}{x(\ln x)^5}$. (Let $u = \ln x$ to make the function $1/u^4 = u^{-4}$.)

(28) $-\sin(\ln(x^3+1))\dfrac{1}{x^3+1}(3x^2) = \dfrac{-3x^2\sin(\ln(x^3+1))}{x^3+1}$. (Let $u = \ln(x^3+1)$ to make the function $\cos u$.)

(29) $3(\ln(\cos x))^2\dfrac{1}{\cos x}(-\sin x) = \dfrac{-3(\ln(\cos x))^2\sin x}{\cos x}$. (Let $u = \ln(\cos x)$ to make the function u^3+1.)

(30) $\dfrac{1}{(2\sin(x^3))^2+1}(2\cos(x^3)(3x^2)) = \dfrac{6x^2\cos(x^3)}{4\sin^2(x^3)+1}$. (Let $u = 2\sin(x^3)$ to make the function $\arctan u$.)