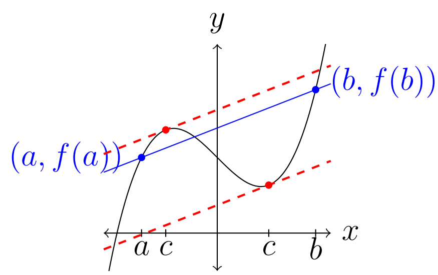
Applications: Mean Value Theorem

When a continuous function $f(x)$ on $[a, b]$ is differentiable on (a, b), the Mean Value Theorem says the average rate of change $(f(b) - f(a))/(b - a)$ on $[a, b]$ is an instantaneous rate of change $f'(c)$ at some c between a and b. In the picture below, the line connecting $(a, f(a))$ and $(b, f(b))$ has its slope equal to the tangent line slope on the graph at two numbers c (not just one) between a and b.



Does this technical-sounding property have a practical use? Here is one.

Application 1. Suppose a car travels along a 5-mile route in 4 minutes. Then its *average* speed over this time interval is

$$\frac{5\,\text{mi}}{4\,\text{min}} = \frac{5\,\text{mi}}{4\,\text{min}}\frac{60\,\text{min}}{1\,\text{hr}} = \frac{300}{4}\,\frac{\text{mi}}{\text{hr}} = 75\,\text{mph}.$$

A car's position $s(t)$ is a continuous function of time, and since velocity is $s'(t)$, it is reasonable to say $s(t)$ is a differentiable function of time. So the Mean Value Theorem says an average velocity of 75 mph over 4 minutes is an instantaneous velocity at some time during the 4 minutes of travel: the car at least once had a speed of 75 mph.

Here is how this can be used. Set up cameras at widely separated positions on a road and detect the same car at different positions (and times) by matching license plates. Compute the car's average velocity between the positions, and this is an instantaneous velocity at some time in between thanks to the Mean Value Theorem: a car with an average velocity of 75 mph between two times was going at 75 mph at some intermediate time. Send the car's owner a speeding ticket for driving at 75 mph even without knowing at what moment that happened.

This is the idea behind *average speed camera systems*, which have been set up in in the UK and Australia. They are not used in the US or Canada yet.

To avoid getting caught by a single speed-detecting camera you need to be within the speed limit only when you're near that camera, but to avoid getting caught by an average speed camera system, the Mean Value Theorem says you need to be within the speed limit during the entire time time you're between two of its cameras. That is why a news headline once said "Canadian Average Speed Cameras Would Make It Impossible For You To Speed And Get Away With It". That is not *quite* true: a driver caught speeding by such a system could dispute a ticket in court by saying "The police can't show that I was always driving at a differentiable function of time," but the judge will not understand this objection.

Except for that previous application, other uses of the Mean Value Theorem are conceptual: it is used in the proofs of theorems, as in the next two applications.

Application 2. Justifying certain properties of derivatives.

- (Section 4.2) when $f'(x) = 0$ on an interval, $f(x)$ is constant there, and when $f'(x) = g'(x)$ on an interval, $f(x) - g(x)$ is constant there,
- (Section 4.3) the first and second derivative tests,
- (Section 4.4) L'Hospital's rule.

The Mean Value Theorem also occurs in the proofs of results that are taught in later math courses, *e.g.*, equality of mixed partial derivatives (Math 2110) and the existence/uniqueness theorem for first-order differential equations (Math 2410).

Application 3. Here is a property of exponential values that isn't about calculus. If t is 0 or a positive integer, then all the numbers in the list

$$2^t,\ 3^t,\ 4^t,\ 5^t, \ldots \tag{1}$$

are integers. Is the reverse direction true?

If each number in (1) is an integer then it can be shown that, indeed, t must be 0 or a positive integer. The proof uses the Mean Value Theorem: see p. 275 of *The American Mathematical Monthly* **80** (1973).[1]

Note. It turns out that if we only know that 2^t, 3^t, and 5^t are integers then t still must be 0 or a positive integer, but this is *much* more difficult. The same conclusion is expected knowing only that 2^t and 3^t are integers, but that is an unsolved problem.

[1] In `https://kconrad.math.uconn.edu/blurbs/analysis/MVT-integral-powers.pdf` a proof is given with more details.