Math 1131 Applications: Exponential Growth/Decay

The most important use of derivatives in applications of calculus is the description of dynamically changing quantities by differential equations. A differential equation involves an unknown function and its derivatives. Examples include $y'(t) = 3y(t)$ and $y''(t) = y(t)^2 - y(t)$. In practice the variable t is usually time. People who care about solving differential equations are interested in both approximating solutions (with a computer) and qualitative features of solutions, *e.g.*, will solutions blow up in finite time? There is a million dollar prize for understanding the solutions to one particular differential equation.

The scope of applications of differential equations, illustrated below, is vast.

- Physics: gravitational, nuclear, and electromagnetic forces
- Engineering: vibrations of mechanical systems, heat flow, electrical circuits
- Chemistry: chemical concentrations during a reaction, molecular interactions
- Biology: spread of an infection or pandemic, metabolism, population growth
- finance: movement of stock prices, pricing insurance products

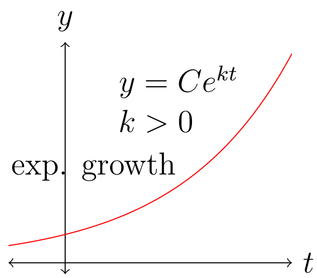
This just scratches the surface. A list of named differential equation is here.

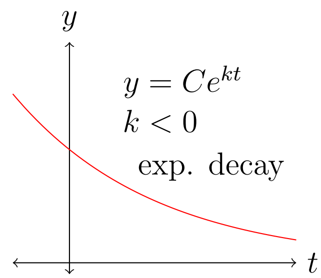
Remark. Computer simulation software for physical systems hides the underlying math, so the following remark taken from here is worth keeping in mind: "Perhaps the reason why some engineers and engineering students feel differential equations are not used by engineers is that they are working with simulating and modeling software [...] and don't see the actual mathematical model behind them."

The most basic widely applicable differential equation is $y'(t) = ky(t)$ for a constant k, and its general solution is

$$y(t) = Ce^{kt}$$

where $C = y(0)$ (the initial amount of y). The numbers C and k are constants, with C usually positive. If $k > 0$ these solutions describe exponential growth, and if $k < 0$ these solutions describe exponential decay. See the graphs below.





Examples of exponential growth include

- the size of a population with no predators or other factors restricting its size[1],
- the amount of money in an account subject to compound interest, particularly continuously compounded interest,
- a nuclear chain reaction.

Examples of exponential decay are

- the concentration of a drug in the blood after it is no longer being administered,
- atmospheric pressure as a function of height above sea level,
- the amount of remaining radioactive atoms in a pile.

A controlled use of exponential growth is how nuclear power plants work and is one reason nuclear bombs are hard to construct. This involves the proper handling of prompt and delayed neutrons. Improper handling is one cause of accidents at nuclear power plants.

Two Nobel prizes have been awarded for research involving physical quantities fitting the differential equation $y'(t) = ky(t)$:

1. 1960 Nobel Prize in Chemistry to Libby for his creation of radiocarbon dating. This is a method of determining the age of old organic material such as prehistoric cave paintings and parchment manuscripts by measuring its carbon isotopes, one of which is subject to radioactive (exponential) decay.

[1] When there are predators, the population size both grows and decays. A basic model for this is the Lotka–Volterra equations.

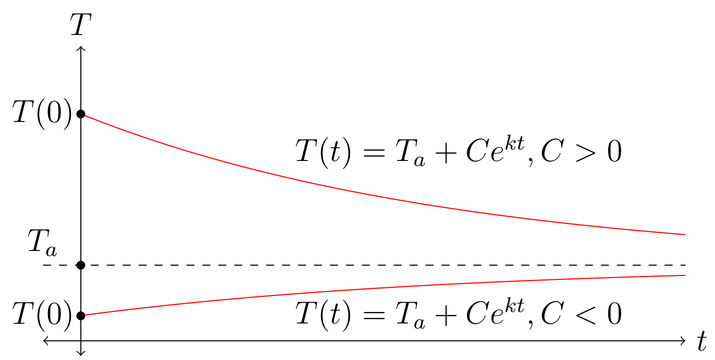
2. 1979 Nobel Prize in Physiology or Medicine to Cormack and Hounsfield for their independent work on CT scanning. Cormack's mathematical work used the Beer–Lambert law, which comes from solving a differential equation that includes $y'(t) = ky(t)$ as a special case.

Every substance undergoing exponential decay has a half-life: the time needed for an amount of the substance to decay to half its value. For example, Gold-238 has a half-life of 2.7 days while Carbon-14 has a half-life of around 5730 years. A modern experimental method to measure half-life is scintillation counting. Half-life can be related to "k" in the exponential decay formula $y(t) = Ce^{kt}$ with $k < 0$: the time $t_{1/2}$ such that $y(t_{1/2}) = y(0)/2 = C/2$ satisfies $C/2 = Ce^{kt_{1/2}}$, so $e^{-kt_{1/2}} = 2$ (the factor $C = y(0)$ has canceled out). Taking natural logarithms of both sides, $-kt_{1/2} = \ln 2$, so $t_{1/2} = (\ln 2)/(-k) = (\ln 2)/|k|$.

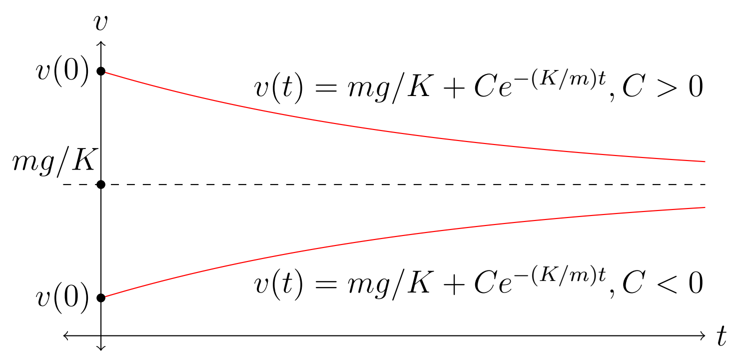
Some differential equations that are not directly of the form $y'(t) = ky(t)$ for constant k can be solved in a similar way to that equation, such as $\boxed{y'(t) = k(y(t) - b)}$ for constants k and b. Rewrite it as $(y(t) - b)' = k(y(t) - b)$, so $y(t) - b = Ce^{kt}$ for a constant C, so $\boxed{y(t) = b + Ce^{kt}}$. (Here C is *not* $y(0)$, but $y(0) - b$.) If $k < 0$ then $y(t) \to b$ as $t \to \infty$, so b is the "terminal" (limiting) value of $y(t)$ for large t. Think of a hot object cooling down to room temperature, a cool object warming up to room temperature, or a falling object reaching terminal velocity. Here are examples of this differential equation in action.

- Newton's law of cooling for an object placed in a large room says its temperature decays at a rate proportional to the difference between its current temperature and the ambient (room) temperature: $T'(t) = k(T(t) - T_a)$, where $k < 0$ and T_a is the ambient temperature. This is the boxed differential equation above, with $b = T_a$. Then $T(t) = T_a + Ce^{kt}$. From $k < 0$, $T(t) \to T_a$ as $t \to \infty$, so the object's temperature tends to room temperature, a familiar physical result. For $T(0) > T_a$ we have cooling, and for $T(0) < T_a$ we have warming. See the first graph below. Newton's law of cooling is a good approximation if the object's initial temperature $T(0)$ is within 50° F of T_a. (This validity when $|T(0) - T_a|$ is small enough is like $\sin\theta \approx \theta$ for small θ in radians.)

- A model for an object moving through air subject to air drag: if air drag is proportional to velocity, which is a good approximation for falling mist particles of oil or water, then by Newton's second law, the object's velocity $v(t)$ satisfies the equation $mv'(t) = mg - Kv(t)$, where m is the mass, $g \approx 9.8\text{m/s}^2$, and the constant $K > 0$ depends on physical properties of the object and air. That equation is the same as $v'(t) = -(K/m)(v(t) - mg/K)$, which is the boxed differential equation above with $k = -K/m$ and $b = mg/K$. Thus $v(t) = mg/K + Ce^{-(K/m)t}$ for a constant C. The terminal velocity (limit of $v(t)$ as $t \to \infty$) is $b = mg/K$. See the second graph below. This was used in Millikan's oil drop experiment, which was the first measurement of an electron's charge and won Millikan the 1923 physics Nobel prize. In the experiment, the drag force is from Stokes' law and depends on the oil's viscosity and droplet radius.



Temperature subject to Newton's Law of Cooling



Velocity when air drag is proportional to velocity

(For a falling skydiver before the parachute opens or a falling penny, air flow is turbulent and air drag is proportional to v^2, not v, so $mv'(t) = mg - Kv(t)^2$ for a constant $K > 0$. Thus $v'(t) = -(K/m)(v(t)^2 - mg/K)$. Solving this differential equation uses methods from Math 1132. Terminal velocity is now $\sqrt{mg/K}$ instead of mg/K: much smaller! For example, 200 mph is replaced by $\sqrt{200} \approx 14$ mph. Videos discussing this without calculus are here and here.)

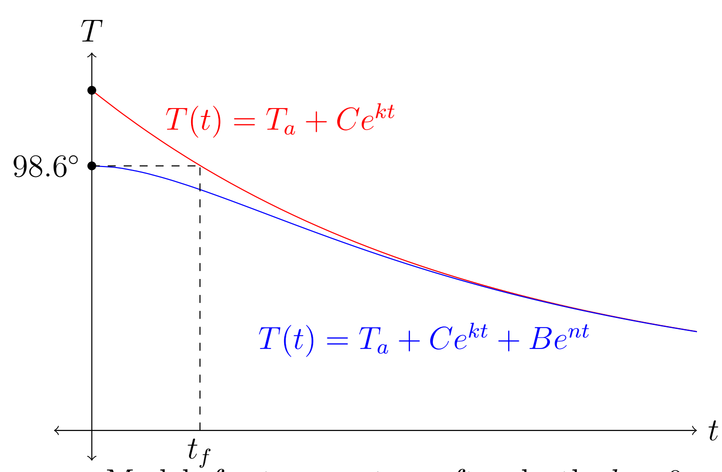
A morbid application of Newton's law of cooling is "time since death": the temperature of the human body after death decreases (if its temperature at death is above the ambient temperature), so if we have a differential equation to model the temperature over time, then solving it and extrapolating backwards in time to when the temperature was 98.6° F (normal human body temperature) can give an estimate for the time of death. This idea arose in the 1800s, so it was not due to Newton.

Let $T(t)$ be the temperature of a body at time t after death. If T were to decay according to Newton's law of cooling, so $T'(t) = k(T(t) - T_a)$ where $k < 0$, then $T(t)$ would be a shifted decaying exponential function like the red curve below. Scientific studies have shown body temperature after death does not decay like an exponential function, but instead is initially relatively flat before starting to decay exponentially, like the blue curve below. Extrapolating backwards to $T = 98.6°$ using the red curve suggests a false time of death t_f that is late. How can we adjust the mathematical model to match the blue curve?

In 1962, Marshall and Hoare proposed adding an additional exponential term to the right side of the differential equation:

$$T'(t) = k(T(t) - T_a) + \ell e^{nt}$$

for constants ℓ and n with $n < 0$. This first-order equation can be turned into a second-order equation $T''(t) - (k+n)T'(t) + knT(t) = knT_a$ (eliminating ℓ). Solutions to this equation are a sum of two exponentials (and a shift): $T(t) = T_a + Ce^{kt} + Be^{nt}$. That is a better fit to the blue curve, For what coroner's *really* do, look here.



Models for temperature after death, $k < 0$