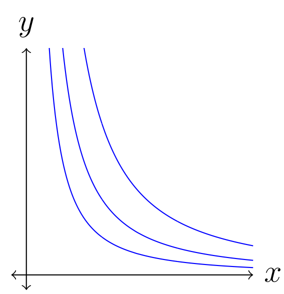
Math 1131 Applications: Implicit Differentiation

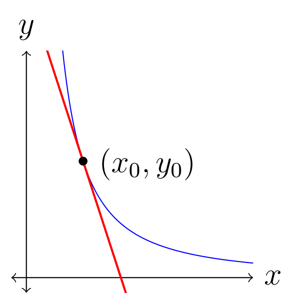
We describe here two applications of calculus where implicit differentiation occurs.

Application 1. Indifference curves in economics

Different amounts of two goods, say apples and oranges, can give a consumer equal satisfaction or, as economists say, the same "utility". If amounts yielding the same satisfaction[1] are plotted on the same curve, then we get what is called an indifference curve: a consumer is indifferent to the specific amounts of each good when the points representing the two amounts of the goods lie on the same indifference curve, since such amounts give the consumer equal satisfaction. Examples of indifference curves corresponding to different levels of a consumer's satisfaction are in the graph below, where x and y are the possible amounts of the two goods that a consumer has.



Each point (x, y) is called a "bundle" of the two goods. If we move from one bundle to another on the same indifference curve, x going up is the same as y going down and x going down is the same as y going up. The slope of a tangent line to an indifference curve at a particular bundle (x_0, y_0) (see picture below) is interpreted as the *rate* at which y needs to change into, or be substituted by, x to give the same level of satisfaction. This slope, or rather its absolute value since the slope is negative, is called the *marginal rate of substitution* (abbreviated as MRS) at (x_0, y_0).



[1]Ignore the popular adage that you can't compare apples and oranges.

To quantify this, we'll consider a common type of indifference curve: $x^a y^b = c$ for positive constants a, b, and c. What is the marginal rate of substitution $|dy/dx|$? Using implicit differentiation,

$$x^a y^b = c \Longrightarrow \frac{d}{dx}(x^a y^b) = \frac{d}{dx}(c) \Longrightarrow a x^{a-1} y^b + x^a b y^{b-1} \frac{dy}{dx} = 0.$$

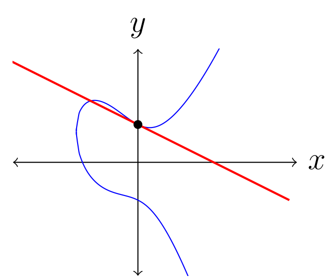
After some algebra,

$$\frac{dy}{dx} = \frac{-a x^{a-1} y^b}{x^a b y^{b-1}} = -\frac{ay}{bx}.$$

Therefore $\boxed{|dy/dx| = ay/bx = (a/b)(y/x)}$.

Application 2. Tangent lines and cryptography.

Below is the graph of $y^2 + xy = x^3 + x^2 + 1$. The point $(0, 1)$ is on this curve. What is the tangent line to the curve at $(0, 1)$?



Using implicit differentiation to differentiate both sides of the equation of the curve with respect to x,

$$\begin{aligned} y^2 + xy = x^3 + x^2 + 1 \quad &\Longrightarrow \quad \frac{dy}{dx}(y^2) + \frac{d}{dx}(xy) = 3x^2 + 2x \\ &\Longrightarrow \quad 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 3x^2 + 2x \\ &\Longrightarrow \quad \boxed{\frac{dy}{dx} = \frac{3x^2 + 2x - y}{2y + x}}. \end{aligned}$$

Therefore

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)} = \frac{3(0)^2 + 2(0) - 1}{2(1) + 0} = -\frac{1}{2}.$$

Thus the tangent line at $(0, 1)$ is the line through $(0, 1)$ with slope $-1/2$, which is $y - 1 = -(1/2)(x - 0)$, or $y = -(1/2)x + 1$.

To appreciate the technique of implicit differentiation that we just carried out, suppose you did *not* know that method and wanted to find the slope of the tangent line to $y^2 + xy = x^3 + x^2 + 1$ at the point $(0, 1)$. What could you do? Rewriting the formula for the curve as

$$y^2 + xy - (x^3 + x^2 + 1) = 0,$$

we can view this as a quadratic equation in y and use the quadratic formula to solve for it in terms of x:

$$y = \frac{-x \pm \sqrt{x^2 - 4(-(x^3 + x^2 + 1))}}{2} = \frac{1}{2}\left(-x \pm \sqrt{4x^3 + 5x^2 + 4}\right).$$

This expresses y explicitly as a function of x (the formula $y^2 + xy = x^3 + x^2 + 1$ is called an "implicit" representation in contrast to that explicit representation of y) and we can differentiate it using the chain rule:

$$\boxed{\frac{dy}{dx} = \frac{1}{2}\left(-1 \pm \frac{1}{2\sqrt{4x^3 + 5x^2 + 4}}(12x^2 + 10x)\right)}.$$

Setting $(x, y) = (0, 1)$, we get

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)} = \frac{1}{2}\left(-1 \pm \frac{1}{2\sqrt{4}}(0)\right) = -\frac{1}{2},$$

which is the same answer we found before.

If we use another point on $y^2 + xy = x^3 + x^2 + 1$, say $(x, y) = (-5/4, -3/8)$, then the contrast between the boxed formula on the previous page for dy/dx from implicit differentiation and the boxed formula for dy/dx above from explicit differentiation of y as a function of x is more vivid. By the first formula,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-5/4,-3/8)} = \frac{3(-5/4)^2 + 2(-5/4) - (-3/8)}{2(-3/8) - 5/4} = \frac{41/16}{-2} = -\frac{41}{32}$$

and by the second formula,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-5/4,-3/8)} = \frac{1}{2}\left(-1 \pm \frac{1}{2\sqrt{4(-5/4)^3 + 5(-5/4)^2 + 4}}(12(-5/4)^2 + 10(-5/4))\right).$$

The number under the square root turns out to be 4, so after some algebra

$$\frac{dy}{dx} = \frac{1}{2}\left(-1 \pm \frac{1}{2\sqrt{4}}(25/4)\right) = \frac{1}{2}\left(-1 \pm \frac{25}{16}\right).$$

With the plus sign this is $9/32$ and with the minus sign this is $-41/32$, so you'd have to figure out which sign is needed for the point we're at[2]. (Comparing with the answer from the first method shows the minus sign is needed, but can you explain the minus sign without relying on the first method?) Such sign issues were unnecessary with the formula for dy/dx by implicit differentiation, so the added effort of using both coordinates x and y in an implicit differentiation formula for dy/dx is worth it.

Finding tangent lines to curves like $y^2+xy = x^3+x^2+1$ is not a pointless exercise: this kind of math is used in elliptic curve cryptography (ECC), which is one of the major ways public key cryptography is implemented. It is part of the security for web browsers, Google Pay, Tor, and Bitcoin.

[2]This sign issue in the second formula for dy/dx is related to the curve having *two* points with $x = -5/4$: $(-5/4, -3/8)$ and $(-5/4, 13/8)$. The tangent line at the first point has slope $-41/32$ and the tangent line at the second point has slope $9/32$.