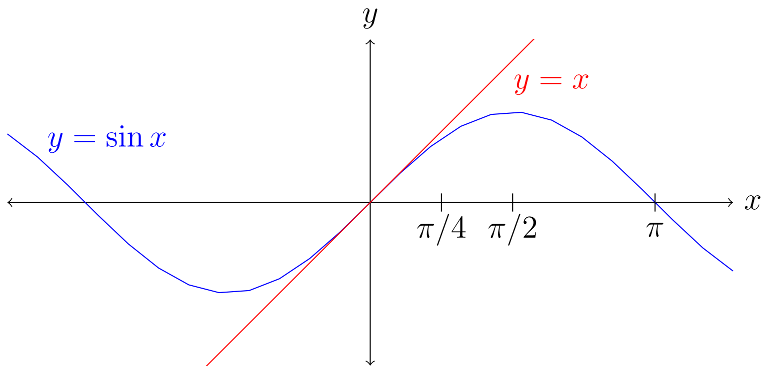
Math 1131 Applications: Small-Angle Approximation

That $\sin'(0) = \cos(0) = 1$ means the tangent line to the graph of $y = \sin x$ at $(0, 0)$ has slope 1: the tangent line is $y = x$. In the picture below, the graph of $y = \sin x$ near $x = 0$ is approximated well by the graph of $y = x$ out to $\pi/4$ radians $= 45°$.



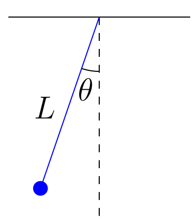
We call $\boxed{\sin x \approx x}$ for small x a *small-angle approximation*. It is illustrated numerically in the table below. The angles are in radians, so $.2 = .2$ radians $\approx 11.4°$. (Multiply by $180/\pi$ to convert from radians to degrees, and by $\pi/180$ to convert from degrees to radians.)

x	.2	.1	.023	.00452	.00059	.000328
$\sin x$	.198669	.099833	.022997	.004519	.000589	.0003279

Continuity of $\sin x$ at $x = 0$ tells us $\sin x \to \sin 0 = 0$ as $x \to 0$. The small-angle approximation for $\sin x$, which is based on differentiability, is an improvement on what we learn from continuity: the small-angle approximation tells us *how* $\sin x$ tends to 0 as $x \to 0$: in a linear (first power) way. Being able to replace the complicated function $\sin x$ with the function x, when x is small, is a convenient approximation in applications.

Application 1. Small oscillations of a pendulum.

If we set a small pendulum in motion, it oscillates back and forth as shown below.



If the pendulum is released *from rest*, then by using Newton's second law (and ignoring friction and air drag) the pendulum's displacement angle $\theta = \theta(t)$ from a vertical position varies with time according to the equation

$$\theta''(t) + \frac{g}{L}\sin\theta(t) = 0 \text{ with } \theta'(0) = 0,$$

where L is the length of the pendulum and $g \approx 9.8$ m/s^2 is the acceleration due to gravity near the surface of the earth. The above equation is analytically hard to solve for $\theta(t)$, but when $\theta(t)$ is small (in radians, so $10^\circ \approx .174$ radians is small) we can approximate the term $\sin\theta(t)$ by $\theta(t)$, which leads to the equation

$$\theta''(t) + \frac{g}{L}\theta(t) = 0 \text{ with } \theta'(0) = 0,$$

and this *can* be solved: $\boxed{\theta(t) = \theta(0)\cos(\sqrt{g/L}t)}$, where $\theta(0)$ is the initial (release) angle for the pendulum at time $t = 0$. (Note $\theta'(t) = -\theta(0)\sin(\sqrt{g/L}t)\sqrt{g/L}$, so $\theta'(0) = 0$, which corresponds to the initial release velocity being 0.) Here are two interesting observations about the boxed formula for $\theta(t)$:

1. Since $\cos x$ has values in $[-1, 1]$, the displacement angle $\theta(0)\cos(\sqrt{g/L}t)$ is never bigger than the release angle $\theta(0)$. See this with a bowling ball pendulum here.

2. The period of $\theta(0)\cos(\sqrt{g/L}t)$ as t varies is[1] $T = 2\pi\sqrt{L/g}$, which is independent of the release angle $\theta(0)$. So the period of a pendulum with different *small* release angles have the same period T. This is shown for a few different angles here. That T does not depend on $\theta(0)$ when $\theta(0)$ is small is the basis for pendulum clocks, which were the primary timekeeping mechanism for over 250 years. If $\theta(0)$ is not small, then T *does* depend on $\theta(0)$: such formulas are here, which are expansions in infinite series (a topic in Math 1132) having $2\pi\sqrt{L/g}$ as the first term.

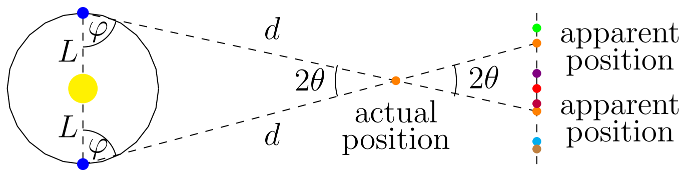
Application 2. Measuring the distance to stars.

The approximation $\sin\theta \approx \theta$ for small θ in radians is the basis for the parallax method of estimating the distance from Earth to the stars.

Stars other than the Sun are so far away that to the naked eye they don't appear to move at all relative to each other over very long periods of time. This is why ancient

[1]The period of $A\cos(c\theta)$, when $A > 0$ and $c > 0$, is $2\pi/c$.

astronomers referred to a background of "fixed stars" against which the planets other than Earth move (the word "planet" is from the Greek term for wanderer: in the night sky, over time, planets move while stars don't). By the 1800s it was possible to detect a *small* apparent motion of some stars relative to the background of other "fixed" stars when observed 6 months apart (meaning the Earth is on opposite sides of the Sun). An analogy that you can check in a room is viewing your finger in front of you with just one eye open and then just the other eye open; your finger has not physically moved, but it will appear to have moved against the background wall (or window, *etc.*).



In the figure above, on the left is the Sun in yellow and the Earth in blue at 2 opposite points in its orbit. Let 2θ be the angle a star in orange appears to sweep out relative to Earth's motion over 6 months (this angle is called the *parallax* of the star[2]). Since 2θ is so small, the triangle connecting the star to the positions of the Earth 6 months apart has two sides of nearly equal length d in the figure, so the triangle is essentially isoceles with the two equal angles φ being nearly $90°$. Thus $\sin\varphi \approx 1$. By the Law of sines, $\sin(2\theta)/(2L) = \sin(\varphi)/d \approx 1/d$. Since 2θ is very small, we can say $\sin(2\theta) \approx 2\theta$, so

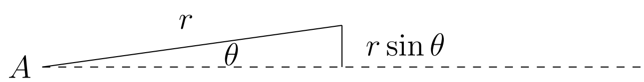
$$\frac{2\theta}{2L} \approx \frac{1}{d} \Longrightarrow d \approx \frac{L}{\theta}.$$

This is how d is estimated. Since θ is extremely small (so small that it couldn't be measured as different from 0 until the 1800s), L/θ is extremely large: astronomical distances are, well, astronomical! How are distances to stars measured when they are so far away that parallax stops working? Find out here.

Application 3. Pilot navigation.

If a pilot intends to fly along a certain straight line route but is off from that direction by a small angle θ, the "1 in 60" rule says that each $1°$ error in direction leads to a 1 mile error from the planned flight path (1 mile "off track") for every 60 miles flown. We'll explain this with the following diagram.

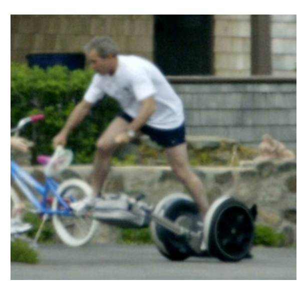
[2] Parallax was used in ancient Greece to measure the distance from Earth to the moon.



If you want to fly from A due east but travel instead at a small nonzero angle θ from an eastern direction, then after traveling r miles the (straight line) distance the plane is from the intended direction is $r \sin \theta$, which for small θ (in radians!) is around $r\theta$. Since $1^\circ = \pi/180$ radians, and $\pi/180 \approx 3/180 = 1/60$, we obtain for $r = 60$ miles and $\theta = 1^\circ$ that $r\theta \approx (60 \text{ miles})(1/60) = 1$ mile. (The actual distance "off track" is $60 \sin(\pi/180) \approx 1.047$ miles.) Some examples aimed at an audience of pilots is here.

Application 4. Stabilizing an unstable system.

The state of a physical system is called *stable* if it is unchanging in time and a small perturbation returns to the original state. An example is a pendulum at rest hanging from the ceiling, as in Application 1. The state of a system is called *unstable* if a small perturbation rapidly moves the system far away from the original state. An example of an unstable state is a person standing up. You don't think of that as unstable, but people who stand make minute corrections automatically, unlike a baby learning to walk or a very drunk person. Stabilizing an unstable state using real-time external feedback is important in technology such as quadcopters and self-balancing transporters (most of the time).



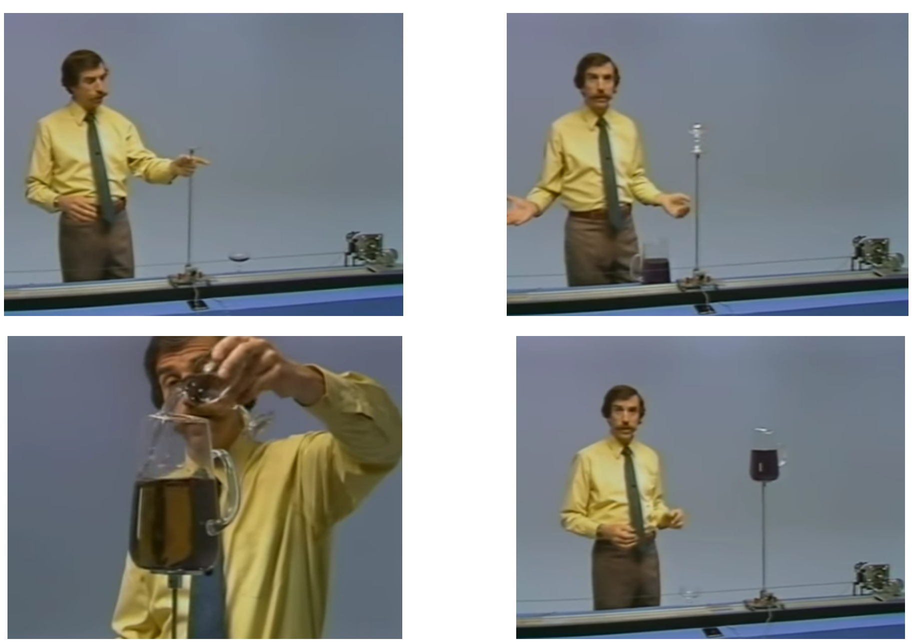
A widely used method of providing external feedback to stabilize an unstable state is PID control, where P stands for proportional and D stands for derivative. The I stands for integral, a calculus topic we'll meet later this semester.

A nice illustration of proportional and derivative control on an unstable state of a system is an MIT video here about an *inverted* pendulum, which is a pendulum that pivots from the bottom rather than the top. The start of the video discusses the underlying math. An equation that models an inverted pendulum's angle from vertical $\theta(t)$, with external feedback, is presented at 6:45:

$$\theta''(t) - \frac{g}{L}\sin\theta(t) = x(t) - \frac{a(t)}{L}\cos\theta(t),$$

where the minus sign in the second term on the left reflects the pendulum being inverted and the terms on the right side are related to external forces. The lecturer uses the small-angle approximations $\sin\theta(t) \approx \theta(t)$ and $\cos\theta(t) \approx 1$ for small $\theta(t)$ (in radians) to simplify the above equation and he then presents a method (called Laplace transforms, beyond the scope of this course) to solve the equation.

The end of the video, from 29:00 onwards, is the best part: watch how the simplified equation is a good enough model to stabilize an inverted pendulum by placing it on a rolling cart. Moving the cart is feedback that counteracts changes in $\theta(t)$. Screenshots from the end of the video are below, showing that added mass such as a cup and a pitcher of water on top of the pendulum don't cause it to tip over. Amazing!



What we need about $\sin x$ to obtain the small-angle approximation $\sin x \approx x$ for small x (in radians) is only two things:

- $\sin(0) = 0$, so the graph of $y = \sin x$ passes through $(0, 0)$,
- $\sin'(0) = 1$, so the tangent line to $y = \sin x$ at $(0, 0)$ is $y = x$.

That $\sin x$ is periodic doesn't matter. *All* functions $f(x)$ with $\boxed{f(0) = 0 \text{ and } f'(0) = 1}$ have tangent line $y = x$ at the origin (see three examples below) and from that we get the good approximation $f(x) \approx x$ for small x.