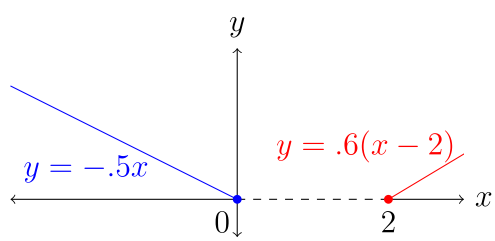
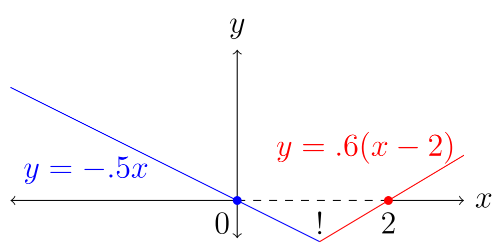
Math 1131 Applications: Curve fitting

Suppose we have the two straight parts of a roller coaster track as shown below.

How can we find a curve connecting them that will give passengers a smooth ride as they transition from one straight track to the other? First let's introduce coordinates so we can describe things with equations, say the following.



We do not want the transition track to be the straight line extensions of each track as shown below: imagine what happens when the roller coaster reaches the corner!



To get a good transition curve we will use a *cubic polynomial*. Set

$$f(x) = \begin{cases} -.5x, & \text{if } x \le 0, \\ ax^3 + bx^2 + cx + d, & \text{if } 0 \le x \le 2, \\ .6(x-2), & \text{if } x \ge 2 \end{cases}$$

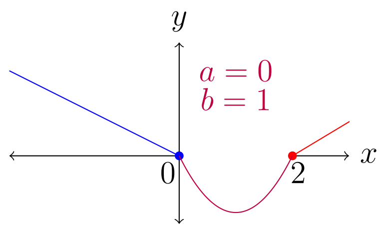
for some a, b, c, and d. (Why not a quadratic polynomial? We'll see later.) In the 2nd picture, we want a **continuous** track: from $f(0) = 0$ and $f(2) = 0$, we need

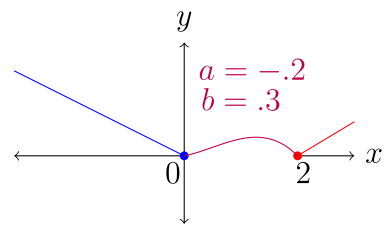
$$\lim_{x\to 0^+} f(x) = 0 \Rightarrow d = 0, \quad \lim_{x\to 2^-} f(x) = 0 \Rightarrow 8a + 4b + 2c = 0 \Rightarrow c = -(4a + 2b).$$

Then $ax^3 + bx^2 + cx + d = ax^3 + bx^2 - (4a + 2b)x = a(x^3 - 4x) + b(x^2 - 2x)$, so

$$f(x) = \begin{cases} -.5x, & \text{if } x \le 0, \\ a(x^3 - 4x) + b(x^2 - 2x), & \text{if } 0 \le x \le 2, \\ .6(x - 2), & \text{if } x \ge 2. \end{cases}$$

Most choices of a and b lead to a bad track: see below. What are good a and b?





To make the transitions in the track at $x = 0$ and $x = 2$ smooth, we want *differentiability* at those points. Derivatives away from $x = 0$ and $x = 2$ are

$$f'(x) = \begin{cases} -.5, & \text{if } x < 0, \\ a(3x^2 - 4) + b(2x - 2), & \text{if } 0 < x < 2, \\ .6, & \text{if } x > 2, \end{cases}$$

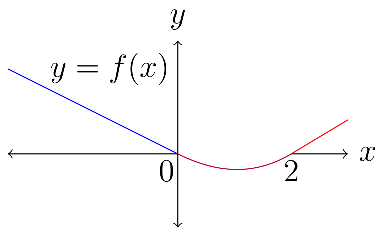
and making limits as $x \to 0^{\pm}$ and as $x \to 2^{\pm}$ agree tells us we want

$$\begin{aligned} -.5 &= -4a - 2b \\ .6 &= 8a + 2b. \end{aligned}$$

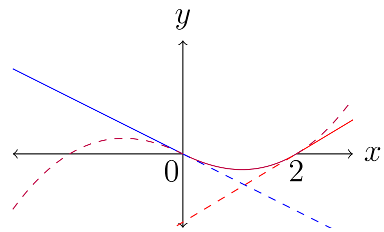
This is two equations in two unknowns, and after some algebra we get the solution $a = .025$ and $b = .2$:

$$f(x) = \begin{cases} -.5x, & \text{if } x \le 0, \\ .025(x^3 - 4x) + .2(x^2 - 2x), & \text{if } 0 \le x \le 2, \\ .6(x - 2), & \text{if } x \ge 2. \end{cases}$$

The left figure below shows how the resulting track appears, and it looks good! The right figure below is the graph for all x of all three formulas making up $f(x)$. At both $(0,0)$ and $(2,0)$ the graphs have *matching tangent lines* from the left and right, which makes the roller coaster track look smooth.



The finished track!



Graphs of all three functions

Now we can explain why we used a cubic polynomial and not a quadratic polynomial for the transition curve. There were four conditions we needed the final curve to satisfy: continuity at both transition points and differentiability at both transition points. A quadratic polynomial $ax^2 + bx + c$ only has 3 coefficients to solve for, which is not enough flexibility when we want the final curve to satisfy 4 constraints.

We only discussed matching function values and first derivatives on a transition curve, but for designs in the real world second derivatives also matter thanks to Newton's second law in physics, which expresses force in terms of acceleration ($F = ma$) and acceleration is a second derivative . Not paying attention to second derivatives can lead to uncomfortable forces at the transition points even if it looks smooth. This matters not only with transition curves for roller coasters, but also for railroad tracks.

Aside from lines transitioning into curves, we may want curves to transition into curves. This is often done with splines, which is a method of curve and surface fitting used in car design and computer graphics, as indicated below (the second image is taken from Terminator 2).

Car design

Computer Graphics

Before computers made spline calculations feasible, automotive designers created transition curves by hand, using a tool with many different curves on it called a

French curve, shown below on the left. The image on the right shows two automotive designers in 1930 and their large supply of French curves.

Another application of curve fitting is the creation of scalable fonts on a computer screen, like the "S" below at two different sizes.

Ss

The next time you zoom in on a web page and the letters on the screen don't pixelate, calculus has a role in that.