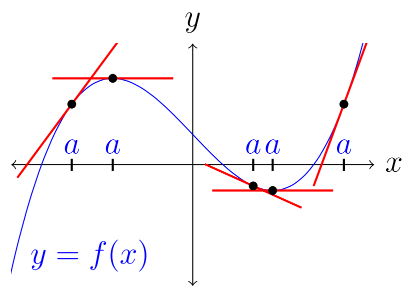
Math 1131 Applications: Derivatives

The derivative of a function $f(x)$ at the number a is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Geometrically this is the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$, and this slope usually varies with a, as in the graph below.



The versatility of the derivative is due to its many interpretations.

- Rate of change: for $x \neq a$, the ratio $(f(x)-f(a))/(x-a)$ is the rate of change of the function over the interval between x and a. Letting $x \to a$ then gives the interpretation of $f'(a)$ as the *instantaneous* rate of change of $f(x)$ at $x = a$ (in applications, x is usually time and written as t). There are many quantities we care about in the world that are dynamic so we care about how they are changing, and this is measured by their rate of change: position (rate of change is velocity), electric charge (rate of change is current), population (rate of change is population growth rate), heat, fluid flow, chemical reactions, drug dosage level in the bloodstream, stock prices, and so on. A 3Blue1Brown video about the instantaneous rate of change is here.

- Relative rate of change: in biology and economics, $f'(x)/f(x)$ is sometimes of greater interest than $f'(x)$ alone. We call $f'(x)/f(x)$ the *relative* rate of change (as a fraction of the current value), although those other disciplines might call $f'(x)/f(x)$ just a "rate of change".

- Linear approximation: for x near a, $f'(a) \approx \dfrac{f(x)-f(a)}{x-a}$. Rearranging the terms, $f(x) - f(a) \approx f'(a)(x-a)$, so $\boxed{f(x) \approx f(a) + f'(a)(x-a)}$. This says that $f(x)$ can be approximated, for x near a, by $f(a)+f'(a)(x-a)$, a *linear polynomial* in x, which is exactly the expression for the tangent line to the graph of $f(x)$ at $(a, f(a))$: $y = f(a)+f'(a)(x-a)$. Writing $x = a+h$, so $x \approx a$ means $h \approx 0$, the linear approximation for $f(x)$ near $x = a$ is the same as $\boxed{f(a+h) \approx f(a)+f'(a)h}$ when h is small. This refines continuity, which only says for small h that $f(a+h) \approx f(a)$: we get a extra term $f'(a)h$ on the right.

Example. If $f(x) = x^2$ then $f'(a) = 2a$, so in particular $f'(3) = 6$ and $f'(.3) = .6$: for $x \approx 3$, $x^2 \approx 9+6(x-3)$, and for $x \approx .3$, $x^2 \approx .09+.6(x-.3)$. Equivalently, if $h \approx 0$ then $(3+h)^2 \approx 9+6h$ and $(.3+h)^2 \approx .09+.6h$.

Let's see this numerically: if $x = 3.01$ then $x^2 = 9.0601$ and $9+6(x-3) = 9+6(.01) = 9.06$, and if $x = .31$ then $x^2 = .0961$ and $.09+.6(x-.3) = .09+.6(.01) = .096$. Equivalently, when $a = 3$ and $h = .01$, $(a+h)^2 = 9.0601 \approx 9+6h = 9.06$, and when $a = .3$ and $h = .01$, $(a+h)^2 = .0961 \approx .09+.6h = .096$.

- Error propagation: for x near a, the formula $\boxed{f(x)-f(a) \approx f'(a)(x-a)}$ says that the function $f(x)$ approximately scales errors $x-a$ by the numerical factor $f'(a)$, so derivatives help estimate *how much* the function magnifies or shrinks errors when it is applied. (That the derivative is an approximate local scaling factor is explored in a 3Blue1Brown video here.)

  Example. If $f(x) = x^2$, so $f'(3) = 6$ and $f'(.3) = .6$, then for $x \approx 3$ we have $x^2 - 9 \approx 6(x-3)$, and for $x \approx .3$ we have $x^2 - .09 \approx .6(x-2)$. Checking this for $x = 3.01$ we have $x^2 - 9 = .0601$ while $6(x-3) = 6(.01) = .06$, and for $x = .31$, $x^2 - .09 = .0061$ and $.6(x-.3) = .6(.01) = .006$.

- Marginal functions: in economics, let $C(q)$ be the cost of making q widgets ($q =$ "quantity"). The cost of making one additional widget after making q of them is $C(q+1)-C(q)$. This is like a derivative: $C'(q)$ is the limit of $(C(q+h)-C(q))/h$ as $h \to 0$, and at $h = 1$ it's $C(q+1)-C(q)$. If $R(q)$ is the revenue from selling q widgets then the revenue from selling one additional widget after selling q of them is $R(q+1)-R(q)$, which is approximately $R'(q)$. Economists call the cost or revenue associated to one additional item the *marginal cost* or *marginal revenue* and may write $MC(q)$ and $MR(q)$ instead of $C'(q)$ and $R'(q)$. In economics, the term "marginal" is always related to derivatives.