

Math 1131 Applications: Continuity

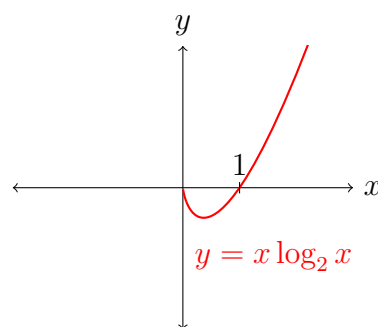
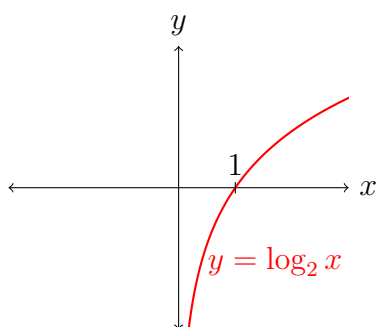
Continuity of a function, which means $\lim_{x \rightarrow a} f(x) = f(a)$ for each a in the domain of $f(x)$, tells us that *nearby inputs in the domain lead to nearby outputs*. This is a feature of most functions you know: polynomials, power functions (like \sqrt{x}), exponential and logarithmic functions, $\sin x$, and $\cos x$ are all continuous. The notion of continuity gives a *name* to this property. Why does this property matter?

- The whole idea of approximate calculations depends on continuity. To estimate $\log_{10} \pi$, you can't plug π exactly into a calculator or computer since they store numbers to a limited amount of decimals, maybe 20 digits, but π is not a 20-digit decimal: its decimal expansion never ends. Since $\log_{10} x$ is continuous, you can *estimate* $\log_{10} \pi$ by using an approximation like 3.14159265 in place of π .
- Equations for how a physical system evolves have parameters, like the initial position and velocity of an object used to predict its future position. In practice parameters (or any measurements) are known only approximately, so we want solutions to the equations to have *continuous dependence on the parameters*.

Example. Solutions of $x^2 - tx + 1 = 0$ are $x = (t + \sqrt{t^2 - 4})/2$ and $x = (t - \sqrt{t^2 - 4})/2$, which are continuous in t (if $|t| \geq 2$, so the solutions make sense in \mathbf{R}). This says solutions of $x^2 - tx + 1 = 0$ depend continuously on t .

Remark. Another feature of interest besides continuous dependence on parameters is sensitive dependence on initial conditions, also known as the [butterfly effect](#). It is studied in [chaos theory](#).

- Continuity can suggest how to define a function at a point where it is not initially defined. Consider the functions $f(x) = \log_2 x$ and $g(x) = x \log_2 x$ when $x > 0$, shown below. Can we extend the functions to $x = 0$?

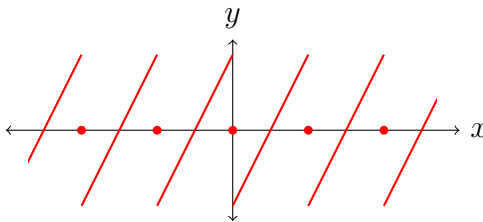


Since $f(x)$ and $g(x)$ are continuous for $x > 0$, the most natural way to extend them to $x = 0$ is by limits: $f(0) := \lim_{x \rightarrow 0^+} \log_2 x$ and $g(0) = \lim_{x \rightarrow 0^+} x \log_2 x$. As the graphs suggest, $\lim_{x \rightarrow 0^+} \log_2 x = -\infty$ and $\lim_{x \rightarrow 0^+} x \log_2 x = 0$. Thus it's natural to set $\log_2 0 = -\infty$ and $0 \log_2 0 = 0$ by “continuity from the right”.

Continuity is the most basic nice property a function can have. There are nicer properties we can hope for, with labels like differentiability, smoothness, and analyticity; you won't see the last two in this course.

We have emphasized the importance of continuity, but discontinuities (or approximations to them) do show up in the real world. Here are examples of jump discontinuities.

1. A [sawtooth wave](#), shown below, is a fundamental example on synthesizers and in signal processing. To hear a sawtooth wave, see the video [here](#).



2. In 2007, F-22s (pictured below on left) [lost their computer systems](#) when crossing the International Date Line since longitude jumps there from -180° to 180° .



3. At the edge of a shock wave (pictured above on right), air pressure *nearly* has a jump discontinuity at macroscopic scales.
4. A [phase transition](#) of matter is often accompanied by a jump discontinuity in a derivative of a thermodynamic parameter (*e.g.*, density or Gibbs free energy).

5. The energy levels of an electron are discrete, *not* continuous. Many physical quantities (energy, charge, light) that appear to vary continuously at ordinary scales are discrete (physicists say “quantized”) at the atomic scale.

6. Videos that you watch present the illusion of continuity but are ultimately discrete (like pixels on a computer screen or 24 frames per second in a movie). Amusing examples where the continuity illusion is broken are the helicopter blades [here](#) and [here](#). Check out the rolling shutter effect [here](#). A presentation about all of these video illusions is [here](#).