Math 1131 Applications: Limits

Passing to a limit is a feature of calculus separating it from algebra. The Up and Atom video here shows how calculus was motivated by 3 paradoxes whose solutions involve limits. Our course could be viewed as one long application of limits. We will focus here on uses of limits that are often not seen in first-semester calculus.

Limits and irrational exponents. In school you learn exponents in stages:

1. An exponent that is a positive integer means repeated multiplication:

$$2^m = \underbrace{2 \cdot 2 \cdots 2}_{m \text{ times}}.$$

   We have the rules $2^m 2^n = 2^{m+n}$ and $(2^m)^n = 2^{mn}$ for positive integers m and n.

2. An exponent that is 0 or a negative integer means: $2^0 = 1$ and $2^{-n} = 1/2^n$ for a negative integer $-n$. For example, $2^{-3} = 1/2^3 = 1/8$. *This is no longer repeated multiplication!* You can't "multiply 2 by itself -3 times."

   With these definitions, the rules $2^m 2^n = 2^{m+n}$ and $(2^m)^n = 2^{mn}$ are valid when m and n are arbitrary integers, even 0 or negative.

3. An exponent that is rational means: $2^{1/q} = \sqrt[q]{2}$ and $2^{p/q} = \sqrt[q]{2^p}$ for positive integers q and any integer p. For example, $2^{1/3} = \sqrt[3]{2}$ and $2^{4/5} = \sqrt[5]{2^4} = \sqrt[5]{16}$.

   With this definition, the rules $2^r 2^s = 2^{r+s}$ and $(2^r)^s = 2^{rs}$ are valid when r and s are arbitrary rational numbers.

   Remark. Rational exponents, usually in the form of finite decimals between 0 and 1, are used in Cobb–Douglas production functions in economics.

So far we have been using algebra. Now allow an irrational exponent, like in 2^π. A calculator says 2^π is $8.8249778\ldots$. *What does that mean?* The number $\pi = 3.14159\ldots$ is the limit of 3, 3.1, 3.14 , 3.141, and so on. Look at this table:

2^3	$2^{3.1}$	$2^{3.14}$	$2^{3.141}$	$2^{3.1415}$	$2^{3.14159}$
8	$8.5741\ldots$	$8.8152\ldots$	$8.8213\ldots$	$8.8244\ldots$	$8.8249\ldots$

The exponents 3, $3.1 = 31/10$, $3.14 = 314/100$, etc. are rational. The powers in the table seem to be heading to a value $8.824\ldots$, and 2^π is defined to be that **limit**: 2^π is the limit of $2^{p/q}$ for rational $p/q \to \pi$. In the same way, 2^x for an irrational number x is the limit of the numbers $2^{p/q}$ for rational $p/q \to x$. With this definition, the rules $2^x 2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$ are valid for all real numbers x and y.

Is this useful? Yes! For example, logarithms can be thought of as exponents ($b^{\log_b x} = x$) and many logarithms (like $\log_{10} 2$) are irrational. There wouldn't be a nice graph for $y = \log_b x$ if b^y didn't have a meaning for irrational exponents y.

Limits and geometry. That the area of a circle is πr^2 is explained using limits by Derek Muller (0:18-1:18), Steven Strogatz, and Grant Sanderson (1:27-6:50). In particular, they use limits to show why the appearance of π in a circle's circumference formula explains the appearance of π in its area formula.

Limits and physics. Two ways in which limits appear in physics is (i) deriving physical laws and (ii) checking the compatibility of new physical theories with older ones under conditions where the old one fits experiments well.

1. Deriving physical laws.

There are many equations in physics telling us how things evolve: the heat equation, the wave equation, and so on. If you look up the *derivation* of such equations by physical reasoning, you will find a process of discretization (small intervals of length or time, say) and then a limit as the discretization tends to 0 ($\Delta x \to 0$ or $\Delta t \to 0$).

2. Compatibility between physical theories.

In the early 1900s, Newton's law of gravity was replaced by Einstein's relativity theory and classical mechanics was replaced by quantum mechanics. Two features of the new physics was the use of v/c in relativity, where v is an object's speed and c is the speed of light (no physical object can travel at that speed, so $v/c < 1$), and a new physical constant h (Planck's constant) in quantum mechanics. Newtonian physics had been successful for 200+ years before 1900, so there should be a compatibility between the old and new physics under conditions when the old physics was already experimentally well tested.[1] At ordinary scales our speeds are *much less* than the speed of light and Planck's constant in ordinary units is *very small* (around 6.626×10^{-34} Joules-sec), so classical physics can be viewed as a limiting case of modern physics by letting $v/c \to 0$ in relativistic formulas and $h \to 0$ in quantum formulas.

Example 1. If P, Q, and R are three particles traveling along a straight line, v_{PQ} is the velocity of P as measured by Q and v_{QR} and v_{PR} are defined similarly, then classically $v_{PR} = v_{PQ} + v_{QR}$. This fits our intuition – if a car travels at 40 mph on

[1] Classical physics continues to be widely used when relativistic and quantum effects are negligible. We did not need relativity or quantum mechanics to send people to the moon in 1969.

a road parallel to train tracks and a train on the tracks goes at 30 mph as measured by someone in the car then the train's velocity will be measured by someone on the ground as being 70 mph, But in relativity theory, v_{PR} is given by another formula:

$$v_{PR} = \frac{v_{PQ} + v_{QR}}{1 + (v_{PQ}v_{QR}/c^2)}.$$

At normal speeds v_{PQ}/c and v_{QR}/c are nearly 0, so their product $v_{PQ}v_{QR}/c^2$ is nearly 0. The relativistic formula for v_{PR} then has denominator nearly 1, making that formula $v_{PR} \approx v_{PQ} + v_{QR}$, which is essentially the classical velocity formula.

Example 2. Quantum mechanics says matter has wave-like properties: wavelength, interference, *etc.*. The wavelength of matter waves is h/p, where h is Planck's constant and p is the matter's momentum (classically, p is mass times velocity). As $h \to 0$ the wavelength h/p is negligible, so we don't see wave-like properties of bulk matter.

Strictly speaking h is a constant, so it can't literally tend to 0. Another way of describing this situation is that at ordinary scales h/p is negligible since $h \approx 6.626 \times 10^{-34}$ J-s is so small.

Relativity and quantum mechanics have features that can't be described by classical physics, such as spacetime curvature and entangled quantum states, but those effects become negligible in the classical limits $v/c \to 0$ or $h \to 0$.

Remark. Elsewhere in physics, compatibility between classical thermodynamics and statistical mechanics uses the thermodynamic limit, which is a limit at ∞.

Limits and animation. The Numberphile video Math and Movies is an interview with Tony DeRose from Pixar Animation. He shows how Pixar creates smooth curves as limits of polygons (by "splitting and averaging") and an analogue of that for surfaces, as seen below in the passage from left to right. Limits and other calculus tools are used to generate shapes in computer animation and to render suitably realistic motion (for clothing, hair, water, *etc.*)