1. Evaluate the following limit: [1]

$$\lim_{x\to\infty} \frac{\sqrt{x^3+2}}{x}.$$

(A) $+\infty$ (B) $-\infty$ (C) 0

(D) 1 (E) -1

this equals $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^3+2}}{\sqrt{x^2}} = \lim_{x\to\infty} \sqrt{x+\frac{2}{x^2}}$

$\text{"="} \sqrt{\infty + 0} \to +\infty$

2. The function $f(x) = \dfrac{x^2+1}{x^2-4}$ has which of the following? [1]

(A) no vertical or horizontal asymptotes

(B) 1 vertical asymptote and 1 horizontal asymptote

(C) 2 vertical asymptotes and 1 horizontal asymptote

(D) 1 vertical asymptote and 2 horizontal asymptotes

(E) 1 vertical asymptote and no horizontal asymptotes

$f(x) = \dfrac{x^2+1}{(x+2)(x-2)} \to$ 2 Vert. asym @ $x = \pm 2$

$\displaystyle\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1+\frac{1}{x^2}}{1-\frac{4}{x^2}} = \frac{1+0}{1-0} = 1$

$\to$ 1 horiz. asym @ $y=1$

3. If $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ for $x > 0$, then $f'(4)$ is which of the following? [1]

(A) $\dfrac{5}{4}$ (B) $\dfrac{3}{4}$ $\boxed{\text{(C) } \dfrac{3}{16}}$

(D) $\dfrac{255}{32}$ (E) $\dfrac{257}{32}$

$$f(x) = x^{1/2} + x^{-1/2} \rightarrow f'(x) = \frac{1}{2}x^{-1/2} + \left(-\frac{1}{2}\right)x^{-3/2}$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \frac{1}{x\sqrt{x}}\right)$$

$$= \frac{1}{2} \cdot \frac{x-1}{x\sqrt{x}}$$

$$f'(4) = \frac{1}{2} \cdot \frac{4-1}{4\sqrt{4}} = \underline{\underline{\frac{3}{16}}}$$

4. Determine $f'(1)$ for the function $f(x) = (x^3 - x^2 + 1)(x^4 - x + 2)$.

(A) 3 (B) 0 (C) 4

(D) 2 $\boxed{\text{(E) } 5}$

$$f'(x) = (3x^2 - 2x)(x^4 - x + 2) + (x^3 - x^2 + 1)(4x^3 - 1)$$

$$\rightarrow f'(1) = (3-2)(1-1+2) + (1-1+1)(4-1)$$

$$= (1)(2) + (1)(3) = \underline{\underline{5}}$$

5. Find the equation of the tangent line to the curve $y = \dfrac{x}{x+1}$ at $x = 1$.

(A) $y = \frac{1}{2}$ (B) $y = -\frac{1}{2}x + 1$ (C) $y = \frac{1}{2}x$

(D) $y = -\frac{1}{4}x + \frac{3}{4}$ $\boxed{\text{(E) } y = \frac{1}{4}x + \frac{1}{4}}$

$$\frac{dy}{dx} = \frac{1(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} \rightarrow \text{at } x=1,\ m = y' = \frac{1}{2^2} = \frac{1}{4}$$

$$\underline{\text{point}}: \ y(1) = \frac{1}{1+1} = \frac{1}{2} \rightarrow \left(1, \tfrac{1}{2}\right) = (x_0, y_0)$$

$$y - y_0 = m(x - x_0) \rightarrow y - \frac{1}{2} = \frac{1}{4}(x-1)$$

$$\rightarrow y = \frac{1}{4}x - \frac{1}{4} + \frac{1}{2} = \underline{\underline{\frac{1}{4}x + \frac{1}{4}}}$$

6. If $f(x) = \sin(x)$, determine $f^{(125)}(\pi)$.

(A) 1 $\boxed{\text{(B) } -1}$ (C) 0

(D) 1/2 (E) $\sqrt{2}/2$

$$f^{(4)}(x) = f^{(8)}(x) = \dots = f^{(124)}(x) = \sin x, \text{ so}$$

$$f^{(125)}(x) = \frac{d}{dx}\left[f^{(124)}(x)\right] = \frac{d}{dx}(\sin x) = \cos x.$$

$$f^{(125)}(\pi) = \cos \pi = \underline{\underline{-1}}$$

7. To compute the derivative of $\sin^2 x$ with the chain rule by writing this function as a composition $f(g(x))$, what is the "inner" function $g(x)$?

(A) x (B) x^2 $\boxed{\text{(C) } \sin x}$

(D) $\sin^2 x$ (E) None of the above

$\sin^2 x = (\sin x)^2$, so $g(x) = \sin x$,
$f(x) = x^2$

8. Let $y = f(x)g(x)$. Using the table of values below, determine the value of $\frac{dy}{dx}$ when $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	5	2	4	4
2	3	4	1	3
3	2	3	2	2
4	4	1	5	5
5	1	5	3	1

(A) 9 (B) 12 $\boxed{\text{(C) } 13}$

(D) 15 (E) 23

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

at $x = 2$: $\frac{dy}{dx} = f'(2)g(2) + f(2)g'(2)$
$= (4)(1) + (3)(3)$
$= 4 + 9 = \underline{\underline{13}}$

9. If $g(x) = \dfrac{ax+b}{cx+d}$, then $g'(1)$ is which of the following? Note: The numbers a, b, c, and d are constants.

(A) $\dfrac{a+b-c-d}{c+d}$ $\boxed{\text{(B) } \dfrac{ad-bc}{(c+d)^2}}$ (C) $\dfrac{a+b-c-d}{(c+d)^2}$

(D) $\dfrac{ad+bc}{c+d}$ (E) $\dfrac{ad+bc}{(c+d)^2}$

$$g'(x) = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2}$$

$$= \frac{acx + ad - (acx + bc)}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2} \rightarrow g'(1) = \underline{\underline{\frac{ad-bc}{(c+d)^2}}}$$

10. For the function $f(x) = x^3 \arctan(x)$, which of the following is $f'(1)$?

(A) $\dfrac{3\pi}{4}$ $\boxed{\text{(B) } \dfrac{3\pi}{4} + \dfrac{1}{2}}$ (C) $\dfrac{1}{2}$

(D) $\dfrac{\pi}{4}$ (E) $3\tan(1) + \sec^2(1)$

$$f'(x) = 3x^2 \arctan(x) + x^3 \cdot \frac{1}{1+x^2}$$

$$\rightarrow f'(1) = 3(1)^2 \arctan(1) + 1^3 \cdot \frac{1}{1+1^2} = \underline{\underline{3\left(\frac{\pi}{4}\right) + \frac{1}{2}}}$$

11. Consider the functions $f(x) = \sin(x^2)$ and $g(x) = \sin^2(x)$. Which of the following is true?

~~(A)~~ $f'(x) = \cos(x^2)$ ~~(B)~~ $g'(x) = -2\sin(x)\cos(x)$ ~~(C)~~ $f'(x) = g'(x)$

~~(D)~~ $f'(\pi) = g'(\pi) = 0$ $\boxed{\text{(E) } f'(0) = g'(0)}$

$f'(x) = 2x\cos(x^2) \neq \cos(x^2) \rightarrow \cancel{A}$

$f'(\pi) = 2\pi\cos\pi^2 \neq 0 \rightarrow \cancel{D}$

$g'(x) = 2\sin x\cos x \neq -2\sin x\cos x \rightarrow \cancel{B}$

not equal $\rightarrow \cancel{C}$

$f'(0) = 2(0)\cos(0^2) = 0 \leftarrow$ E ✓

$g'(0) = 2\sin(0)\cos(0) = 2(0)(1) = 0$

12. If $\dfrac{d}{dx}[f(4x)] = x^2$, then find $f'(x)$.

$\boxed{\text{(A) } \dfrac{x^2}{64}}$ (B) $\dfrac{x^2}{16}$ (C) $\dfrac{x^2}{4}$

(D) x^2 (E) $4x^2$

$\dfrac{d}{dx}[f(4x)] = f'(4x)\cdot 4 = x^2 \rightarrow f'(4x) = \dfrac{x^2}{4}$

Let $u = 4x$. Then $\dfrac{u}{4} = x$, so

$f'(u) = \dfrac{\left(\frac{u}{4}\right)^2}{4} = \dfrac{\frac{u^2}{16}}{4} = \dfrac{u^2}{64}.$

replacing u with x to represent the function,

$f'(x) = \dfrac{x^2}{64}$

13. Find an equation of the tangent line to the curve $(x^2 + y^2)^2 = 4x^2y$ at the point $(1, 1)$.

$\boxed{\text{(A) } y = 1}$ (B) $y = x$ (C) $y = 2x - 1$

(D) $y = -x + 2$ (E) $y = -2x + 3$

$$\frac{d}{dx}\left[(x^2+y^2)^2\right] = \frac{d}{dx}\left[4x^2y\right]$$

$$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 8xy + 4x^2\frac{dy}{dx}$$

at $x=1, y=1$:

$$2(1+1)\left(2+2\frac{dy}{dx}\right) = 8 + 4\frac{dy}{dx}$$

$$4\left(2+2\frac{dy}{dx}\right) = 8+4\frac{dy}{dx}$$

$$8+8\frac{dy}{dx} = 8+4\frac{dy}{dx} \rightarrow 4\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = 0$$

point $(1,1)$, slope $m=0$

$y-1 = 0(x-1)$

$\underline{\underline{y=1}}$

14. Find $\dfrac{d}{dx}\left[\sin(\ln x^2)\right]$.

(A) $\dfrac{-\cos(\ln(x))}{x^2}$ (B) $\dfrac{-2\sin(\ln(x^2))}{x^2}$ (C) $\dfrac{\cos(\ln(x))}{2x^2}$

$\boxed{\text{(D) } \dfrac{2\cos(\ln(x^2))}{x}}$ (E) None of the above

$$\frac{d}{dx}\left[\sin(\ln x^2)\right] = \cos(\ln x^2)\cdot\frac{d}{dx}\left[\ln(x^2)\right]$$

$$= \cos(\ln x^2)\cdot\frac{1}{x^2}\cdot\frac{d}{dx}(x^2)$$

$$= \cos(\ln x^2)\cdot\frac{1}{x^2}\cdot 2x$$

$$= \frac{2x\cos(\ln x^2)}{x^2} = \underline{\underline{\frac{2\cos(\ln(x^2))}{x}}}$$

15. Find $\dfrac{d}{dx}[\log_4(3x)]$.

(A) $\dfrac{1}{3x\ln 4}$ $\boxed{\text{(B) } \dfrac{1}{x\ln 4}}$ (C) $\dfrac{1}{x}$

(D) $\dfrac{3}{x\ln 4}$ (E) $\dfrac{3}{x}$

$$\frac{d}{dx}\left[\log_4(3x)\right] = \frac{1}{3x\ln 4}\frac{d}{dx}(3x) = \frac{3}{3x\ln 4} = \underline{\underline{\frac{1}{x\ln 4}}}$$

16. The size of a colony of bacteria at time t hours is given by $P(t) = 100e^{kt}$, where P is measured in millions. If $P(5) > P(0)$, then determine which of the following is true.

✓ I. $k > 0$

~~II.~~ $P'(5) < 0$

✓ III. $P'(10) = 100ke^{10k}$

$\boxed{\text{(A) I and III only.}}$ (B) I and II only. (C) I only.

(D) II only. (E) I, II, and III.

I. P increasing, so $k > 0$ in $P = P(0)e^{kt}$ ✓

II. $P'(t) = kP(t) > 0$ since $k > 0$ and $P > 0$
$\to P'(5) > 0$ ✗

III. $P'(t) = 100ke^{kt} \to P'(10) = 100ke^{10k}$ ✓

17. Suppose that the half-life of a certain substance is 20 days and there are initially 10 grams of the substance. The amount of the substance remaining after time t is given by

(A) $10e^{10k}$ (B) $\ln(10)e^{kt/10}$ (C) $\ln(10)e^{t/10}$

$\boxed{\text{(D) } 10e^{-t\ln(2)/20}}$ (E) $10e^{t\ln(2)/20}$

$y = Ae^{kt}$, $A = y(0) = 10$: $y = 10e^{kt}$

At $t = 20$, $y = 5$: $5 = 10e^{k(20)}$

$\rightarrow \frac{1}{2} = e^{20k} \rightarrow 20k = \ln\frac{1}{2}$

$k = \frac{\ln\frac{1}{2}}{20}$

$= \frac{-\ln 2}{20}$

$\rightarrow y = 10e^{kt} = \underline{10e^{-t(\ln 2)/20}}$

18. Atmospheric pressure (the pressure of air around you) decreases as your height above sea level increases. It decreases exponentially by 12% for every 1000 meters. The pressure at sea level is 1013 hecto pascals. The amount of pressure at any height h is given by,

(A) $1000e^{10h}$ (B) $\ln(1013)e^{kh/12}$ (C) $1013e^{\ln(0.88)/1000}$

(D) $1000e^{-h\ln(2)/20}$ $\boxed{\text{(E) } 1013e^{h\ln(0.88)/1000}}$

$y = Ae^{kh}$, $A = y(0)$ = pressure at $h = 0$ (sea level)

$= 1013 \rightarrow y = 1013e^{kh}$

At $h = 1000$, $y = 0.88 \cdot 1013 = 88\%$ of sea level pressure

$\rightarrow 0.88(\cancel{1013}) = \cancel{1013}e^{k(1000)}$

$0.88 = e^{1000k} \rightarrow 1000k = \ln(.88)$

$\rightarrow k = \frac{\ln(.88)}{1000}$

So $\underline{y = 1013e^{h\ln(0.88)/1000}}$

19. A particle moves along the curve $y = \sqrt[3]{x^4 + 11}$. As it reaches the point $(2, 3)$, the y-coordinate is increasing at a rate of 32 cm/s. Which of the following represents the rate of increase of the x-coordinate at that instant?

(A) 27 cm/s (B) 9 cm/s (C) 27/2 cm/s

(D) 67/4 cm/s (E) None of the above

$y = \sqrt[3]{x^4+11} \rightarrow y^3 = x^4 + 11$

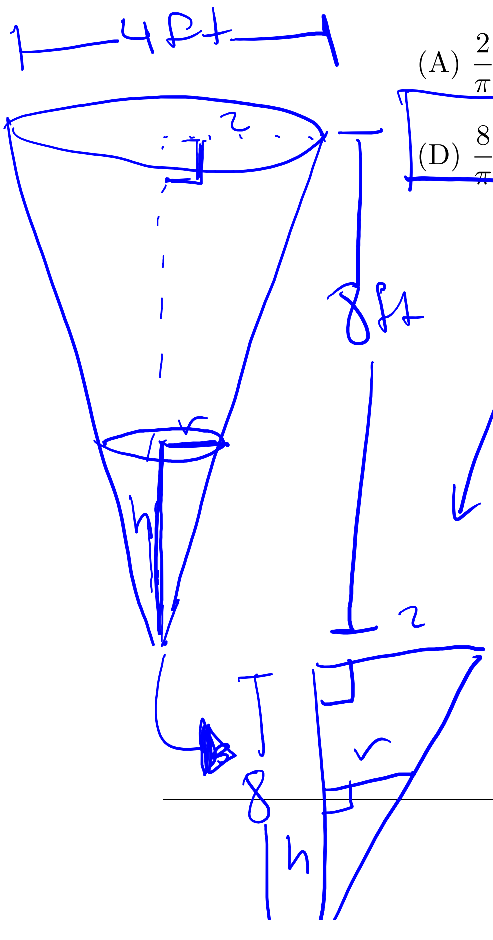
$\frac{d}{dt}(y^3) = \frac{d}{dt}(x^4+11) \rightarrow 3y^2\frac{dy}{dt} = 4x^3\frac{dx}{dt}$

When $x=2$, $y=3$, and $\frac{dy}{dt} = 32$:

$3(3)^2(32) = 4(2)^3\frac{dx}{dt} \rightarrow 27\cdot\cancel{32} = \cancel{32}\frac{dx}{dt}$

$\frac{dx}{dt} = 27$ cm/s

20. Water is withdrawn at a constant rate of 2 ft^3/min from an inverted cone-shaped tank (meaning the vertex is at the bottom). The diameter of the top of the tank is 4 ft, and the height of the tank is 8 ft. How fast is the water level falling when the depth of the water in the tank is 2 ft? (Remember that the volume of a cone of height h and radius r is $V = \frac{\pi}{3}r^2h$?)



(A) $\frac{2}{\pi}$ ft/min (B) $\frac{4}{\pi}$ ft/min (C) $\frac{6}{\pi}$ ft/min

(D) $\frac{8}{\pi}$ ft/min (E) $\frac{16}{\pi}$ ft/min

similar triangles: $\frac{2}{r} = \frac{8}{h} \rightarrow 2h = 8r \rightarrow r = \frac{h}{4}$

Plug $r = \frac{h}{4}$ into Vol. formula:

$V = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3$

$\frac{dV}{dt} = \frac{\pi}{48}\cdot 3h^2\frac{dh}{dt} = \frac{\pi h^2}{16}\frac{dh}{dt}$

when $\frac{dV}{dt} = -2$ and $h = 2$,

$-2 = \frac{\pi(2^2)}{16}\frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{-32}{4\pi} = -\frac{8}{\pi}$

$\rightarrow$ falling @ rate of $\frac{8}{\pi}$ ft/min

21. Determine $f''(x)$ for the function $f(x) = \dfrac{\ln x}{x^2}$.

(A) $\dfrac{-1}{2x^2}$ (B) $\dfrac{6\ln x}{x^4}$ (C) $\dfrac{1-6\ln x}{x^4}$

(D) $\dfrac{1-2\ln x}{x^3}$ $\boxed{\text{(E) None of the above}}$

$$f'(x) = \frac{x^2 \cdot \frac{1}{x} - (\ln x)(2x)}{x^4} = \frac{x - 2x\ln x}{x^4}$$

$$f''(x) = \frac{\left[1-\left(2\ln x + 2x\cdot\frac{1}{x}\right)\right](x^4) - 4x^3\left[x-2x\ln x\right]}{x^8}$$

$$= \frac{x^4\left[1-2\ln x - 2 - 4(1-2\ln x)\right]}{x^8} = \frac{-1-4-2\ln x + 8\ln}{x^4}$$

$$= \frac{-5+6\ln x}{x^4}$$

22. Use the linearization for the function $f(x) = \sqrt{x^3+2x+1}$ at $x=1$ to approximate the value of $f(1.1)$.

(A) $\dfrac{161}{80}$ (B) $\dfrac{21}{10}$ $\boxed{\text{(C) } \dfrac{17}{8}}$

(D) $\dfrac{1}{2}$ (E) $\dfrac{17}{16}$

$$L(x) = f'(1)(x-1) + f(1)$$

$$f(1) = \sqrt{1+2+1} = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2}(x^3+2x+1)^{-1/2}(3x^2+2)$$

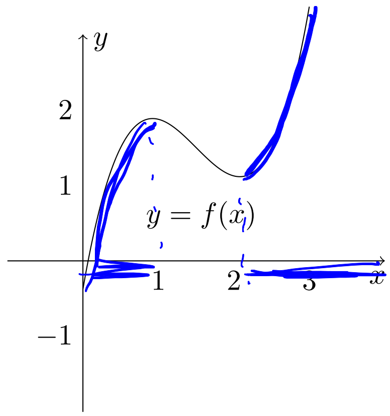
$$f'(1) = \frac{1}{2}(1+2+1)^{-1/2}(3+2) = \frac{1}{2}\left(\frac{1}{2}\right)(5) = \frac{5}{4}$$

$$\rightarrow f(1.1) \approx L(1.1) = f'(1)(1.1-1) + f(1) = \frac{5}{4}(0.1) + 2$$

$$= \frac{5}{40} + 2 = \frac{1}{8} + 2$$

$$= \frac{17}{8}$$

23. The curve below is the graph of $y = f(x)$. List all x-values, in interval form, on which $f'(x)$ (the *derivative* of f) is positive. [1]



(A) $(0, 1)$ (B) $(0, 2)$ (C) $(1, 2)$

(D) $(2, 3)$ (E) $(0, 1)$ and $(2, 3)$

f'(x) is positive when f is increasing

between about x=0 and x=1, and

between about x=2 and x=3.