Section 4.8: Newton's Method

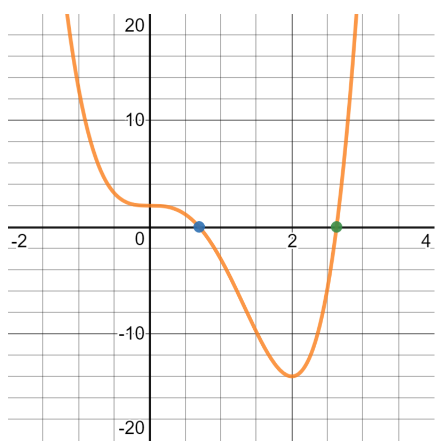
(1) In this section, we talk about Newton's Method. What is Newton's method used for? Newton's method is used to find approximations of the roots of a function, f, or solutions to the equation

$$f(x) = 0.$$

In fact, it gives one an iterative process to find a finite number of approximations of the roots. The goal of Newton's method is find a better approximation of the root than the approximation before it. You may have used your graphing calculator to approximate roots of an equation before. One of the methods that your calculator uses to find these approximations is *Newton's method!*

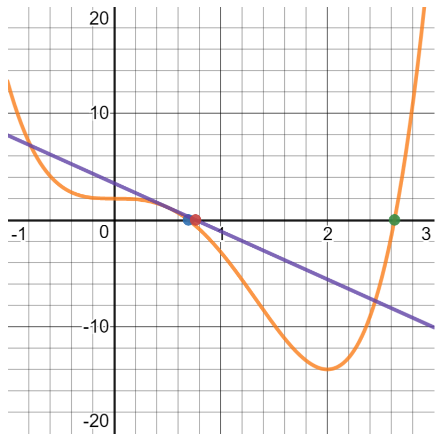
(2) Sketch the graph of a function with at least two zeros. Illustrate Newton's method on your sketch. Show how different starting points can lead to different zeros.

$$f(x) = 3x^4 - 8x^3 + 2$$



First we consider the root given by the blue dot. We guess that it is 0.5, that is, $x_1 = 0.5$. Next we consider the tangent line L_1 (purple) to the curve at the point $(0.5, f(0.5))$ and then find the x-intercept of L_1, which will be x_2. You will see that using fractions in your computations and only rounding a decimal at the end will give you the most accurate approximation.

FIGURE 1. $x_2 \approx 0.7639$ (red dot)



The equation of the tangent line L_1 is given by

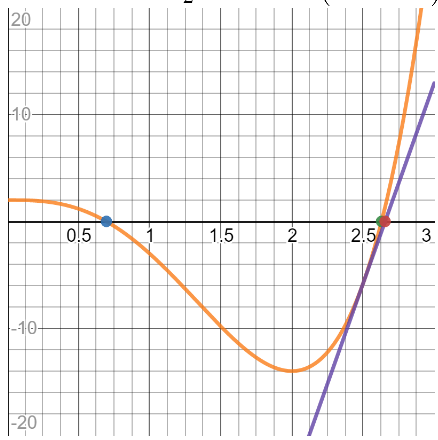
$$\begin{aligned} y &= f'(0.5)(x - 0.5) + f(0.5) \\ &= -\frac{9}{2}(x - 0.5) + \frac{19}{16} \end{aligned}$$

Next we set $y = 0$ and $x = x_2$ and solve for x_2. Thus

$$\begin{aligned} 0 &= -\frac{9}{2}(x_2 - 0.5) + \frac{19}{16} \\ -\frac{19}{16} &= -\frac{9}{2}x_2 + \frac{9}{4} \\ -\frac{55}{16} &= -\frac{9}{2}x_2 \\ \frac{55}{72} &= x_2 \approx 0.7639 \end{aligned}$$

By looking at the graph, we notice that x_2 is a pretty good approximation of the root given by the blue dot. Next we consider the root given by the green dot. We guess that it is $x_1 = 2.5$. Next we consider the tangent line L_2 (purple) to the curve at the point $(2.5, f(2.5))$ and then find the x-intercept of L_2, which will be x_2.

FIGURE 2. $x_2 \approx 2.655$ (red dot)



The equation of the tangent line L_2 is given by

$$\begin{aligned} y &= f'(2.5)(x-2.5)+f(2.5) \\ &= \frac{75}{2}(x-2.5)-\frac{93}{16} \end{aligned}$$

Next we set $y=0$ and $x=x_2$ and solve for x_2. Thus

$$\begin{aligned} 0 &= \frac{75}{2}(x_2-2.5)-\frac{93}{16} \\ \frac{93}{16} &= \frac{75}{2}x_2-\frac{375}{4} \\ \frac{1593}{16} &= \frac{75}{2}x_2 \\ \frac{531}{200} &= x_2 \approx 2.655 \end{aligned}$$

Again we notice that x_2 is a good approximation of the root given by the green dot.

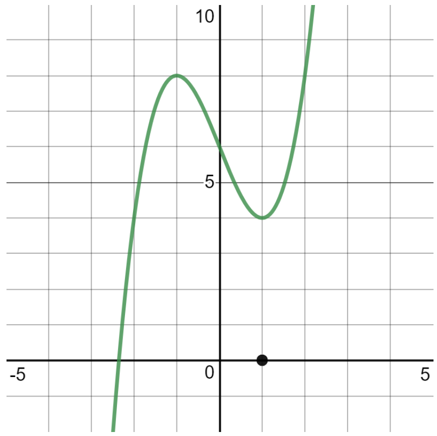
(3) When we are applying Newton's method repeatedly, it is sometimes helpful to remember the formula for the next x_i value. What is that formula? How does it work? Having to repeat the tangent line procedure will be tedious and it will be helpful to remember the formula for the next approximation value. In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(4) What can go wrong if we are trying to do Newton's method? Give an example with both a sketch and a algebraic expression for your function. Be sure to include the starting point.
One issue that may occur when trying to do Newton's method is choosing an initial approximation x_1, such that the derivative at x_1 is zero. Recalling the formula above, we can see how this can be a problem (division by zero). For example, suppose you

have

$$x^3 - 3x + 6 = 0$$



By viewing the graph of $x^3 - 3x + 6$ it is may not seem reasonable to choose $x_1 = 1$, but suppose we did. Then

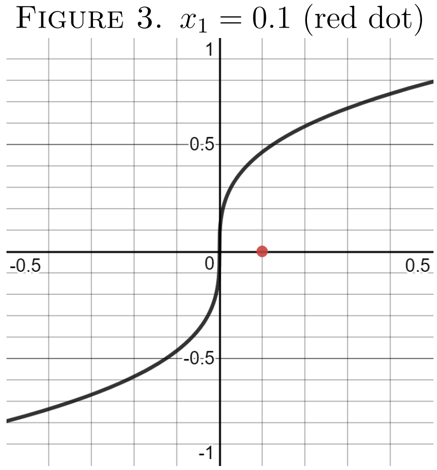
$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{4}{0} \end{aligned}$$

which means we can not proceed with the method.
Another issue that may occur is after choosing your initial approximation, every approximation afterwords is less accurate than the other before it. Consider the equation

$$\sqrt[3]{x} = 0$$

with initial approximation $x_1 = 0.1$.

Figure 3. $x_1 = 0.1$ (red dot)



We see that

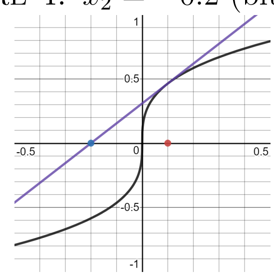
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1 - \frac{f(0.1)}{f'(0.1)}$$

$$= 0.1 - \frac{\sqrt[3]{0.1}}{\frac{1}{3}(0.1)^{\frac{-2}{3}}}$$

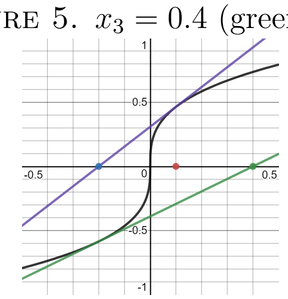
$$\begin{aligned}
&= 0.1 - \frac{\sqrt[3]{0.1}}{\frac{1}{3}(\sqrt[3]{0.1})^{-2}} \\
&= 0.1 - 3\sqrt[3]{0.1}(\sqrt[3]{0.1})^2 \\
&= 0.1 - 3(\sqrt[3]{0.1})^3 \\
&= 0.1 - 3(0.1) = -0.2
\end{aligned}$$

FIGURE 4. $x_2 = -0.2$ (blue dot)



By looking at the graph, we can see that x_2 is a worse approximation than x_1. Similarly we could compute x_3 and see that x_3 is a worse approximation than x_2.

FIGURE 5. $x_3 = 0.4$ (green dot)



Extra Practice in Book: 4.8: 1, 3, 5, 7, 33