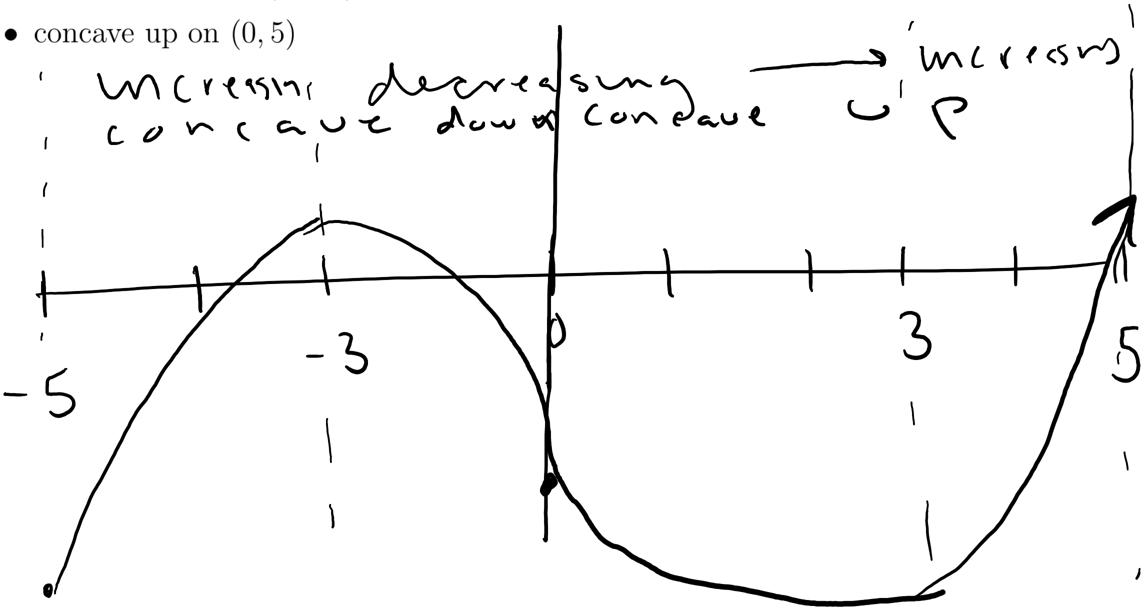
Section 5.2: The Second Derivative
Section Objectives:

- Know how to find the second derivative.
- Know what it means for a functions to be concave up or concave down (both in terms of the first derivative and the graph).
- Know how to use the second derivative to tell if a function is concave up or concave down.
- Know the definition of an inflection point of a function.
- Know how to use the second derivative to determine if a function has a minimum or maximum at a place where the derivative is 0.
- Know how to tell when an economy of scale exists.

Practice Problems

1. Sketch the graph of a function on the domain $[-5, 5]$ which satisfies all the conditions below.
   - increasing on $(-5, -3)$ and $(3, 5)$
   - decreasing on $(-3, 3)$
   - concave down on $(-5, 0)$
   - concave up on $(0, 5)$



2. If $f'(4) = 0$ and $f''(4) = 5$, what can we say about f at $x = 4$? Explain your reasoning.

Since $f''(4) = 5 > 0$ f is concave up at $x=4$ so critical point $(f'(4)=0)$ must correspond to minimum at $x=4$.

3. Let $f(x) = 3x^4 - 4x^3 + 1$. Find the intervals where $f(x)$ is increasing, decreasing, concave up and concave down. Find all relative extrema and inflection points. Use these to sketch a graph of the function.

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$

$f'(x) = 0$ @ $x = 0$, $x = 1$

f': $- \; - \; - \; 0 \; - \; - \; - \; - \; - \; 0 \; + \; + \; + \; + \; +$

f: $\searrow \searrow \searrow \; 0 \rightarrow \; \searrow \searrow \searrow \; 1 \rightarrow \; \nearrow \nearrow \nearrow \nearrow$

f inc $(1, \infty)$ f dec $(-\infty, 1)$ $f(1) = 0$

(no change at $x = 0$), relative min @ $x = 1$ $f(0) = 1$

$$f''(x) = 36x^2 - 24x = 12x(3x-2)$$

$f''(x) = 0$ @ $x = 0, 2/3$

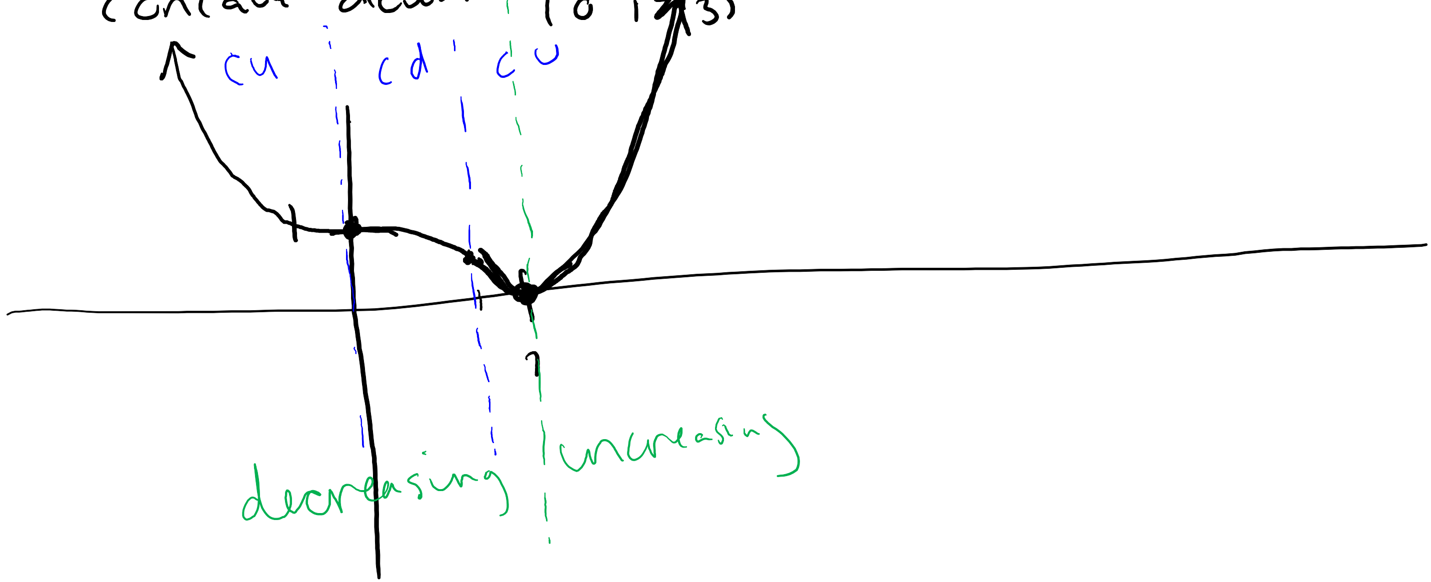
$f(2/3) = .\overline{407}$

f'': $+ \; + \; + \; - \; - \; - \; - \; + \; + \; + \; + \; +$

f: $\cup \cup \cup \; 0 \; \cap \cap \cap \cap \; 2/3 \; \cup \cup \cup \cup$

f concave up $(-\infty, 0) \cup (2/3, \infty)$

concave down $(0, 2/3)$



4. Let $f(x) = e^{-x^2}$. Find the intervals where $f(x)$ is increasing, decreasing, concave up and concave down. Find all relative extrema and inflection points. Use these to sketch a graph of the function.

$f'(x) = e^{-x^2}(-2x)$

$f'(x) = 0$ when $-2x = 0$
$\Rightarrow x = 0$

since $e^{-x^2} \neq 0$.

f' + + + + + 0 − − − − − − −

f ↗ ↗ ↗ ↗ 0 ↘ ↘ ↘ ↘ ↘

f inc $(-\infty, 0)$

f dec $(0, \infty)$

relative max @ 0, $f(0) = 1$

$f''(x) = e^{-x^2}(-2) + e^{-x^2}(-2x)(-2x)$

(product rule)

$= e^{-x^2}(4x^2 - 2)$

$f''(x) = 0$ when $4x^2 - 2 = 0$

since $e^{-x^2} \neq 0$

$\Rightarrow x^2 = 2 \quad x = \pm\sqrt{2}$

f'' + + + 0 − − − − − − − 0 + + + +

∪ ∪ $-\sqrt{2}$ ∩ ∩ ∩ $\sqrt{2}$ ∪ ∪ ∪

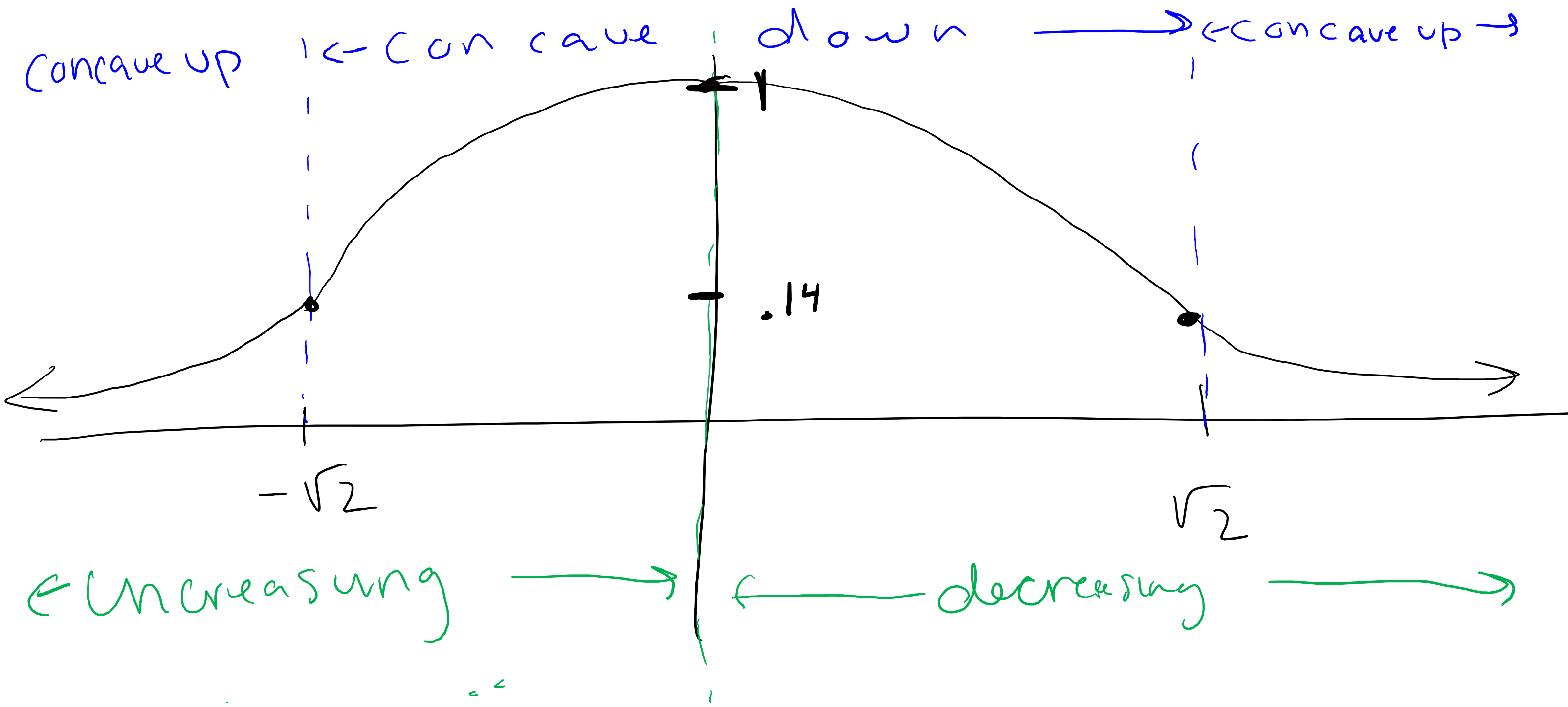
f concave up $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

f concave down $(-\sqrt{2}, \sqrt{2})$

$x = \pm\sqrt{2}$, point of inflection

$f(-\sqrt{2}) = f(\sqrt{2}) \approx .14$

also note: $f(x) > 0$ for all x



5. We have an economy of scale if the marginal cost ($C'(x)$) is decreasing as the number of units produced increases. What does this tell us about $C''(x)$? About $C(x)$?

If $C'(x)$ (marginal cost) is decreasing its derivative $C''(x)$ is negative which means $C(x)$ is concave down when there is an economy of scale.

6. If the cost function of a firm is given by $C(x) = -0.1x^2+2x+5$, is this firm experiencing an economy of scale? Explain your reasoning.

If $C(x) = -.1x^2 + 2x + 5$

then $C'(x) = -.2x + 2$

$C''(x) = -.2 < 0$

so $C''(x) < 0 \Rightarrow C'(x)$ is decreasing

so it is an economy of scale!

More Practice from Textbook 5.2: You should do as many problems from each set (1-6, 7-12, 13-20, 21-24, 25-36, 37-41, 45-62), as needed until you are comfortable with these techniques. 45-62 are good practice for application problems.