

CHAIN RULE PROBLEMS

The chain rule says $(f(g(x)))' = f'(g(x))g'(x)$, or $(f(u))' = f'(u)u'(x)$ if $u = g(x)$. To carry out the chain rule, know basic derivatives well so you can build on that. In the table below, observe how each basic derivative in the first column (based on material through Section 3.6) can be generalized by the chain rule by using $g(x)$ in place of x in the third column.

Basic derivative	Composition	Chain rule example
$(x^c)' = cx^{c-1}$ for constant c	$g(x)^c = u^c$ for $u = g(x)$ and constant c	$(g(x)^c)' = cg(x)^{c-1}g'(x)$ for constant c
$(x^3)' = 3x^2$	$g(x)^3 = u^3$ for $u = g(x)$	$((g(x))^3)' = 3g(x)^2g'(x)$
$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$\sqrt{g(x)} = \sqrt{u}$ for $u = g(x)$	$(\sqrt{g(x)})' = \frac{1}{2\sqrt{g(x)}}g'(x)$
$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$	$\frac{1}{g(x)} = \frac{1}{u}$ for $u = g(x)$	$\left(\frac{1}{g(x)}\right)' = \frac{-1}{g(x)^2}g'(x)$
$(e^x)' = e^x$	$e^{g(x)} = e^u$ for $u = g(x)$	$(e^{g(x)})' = e^{g(x)}g'(x)$
$(\ln x)' = \frac{1}{x}$	$\ln g(x) = \ln u$ for $u = g(x)$	$(\ln g(x))' = \frac{1}{g(x)}g'(x)$
$(\sin x)' = \cos x$	$\sin(g(x)) = \sin u$ for $u = g(x)$	$(\sin(g(x)))' = \cos(g(x))g'(x)$
$(\cos x)' = -\sin x$	$\cos(g(x)) = \cos u$ for $u = g(x)$	$(\cos(g(x)))' = -\sin(g(x))g'(x)$
$(\tan x)' = \sec^2 x$	$\tan(g(x)) = \tan u$ for $u = g(x)$	$(\tan(g(x)))' = \sec^2(g(x))g'(x)$
$(\arctan x)' = \frac{1}{x^2 + 1}$	$\arctan(g(x)) = \arctan u$ for $u = g(x)$	$(\arctan(g(x)))' = \frac{1}{g(x)^2 + 1}g'(x)$

Problems.

- (1) $\sin^2 x$ (This means $(\sin x)^2$, so the “outer” function is x^2 , not $\sin x$.)
- (2) $\sin(\sin x)$
- (3) $\sin(\sin(\sin x))$
- (4) $\sin(\sin(\sin(\sin x)))$
- (5) $\sqrt{x^2 + 5x}$
- (6) $(x^3 + x^2 + 1)^4$
- (7) $e^{-x^2/5}$
- (8) $(\sqrt{x} + x^3)^8$
- (9) $\frac{1}{\cos x}$
- (10) $\cos\left(\frac{1}{x}\right)$
- (11) $3^{\cos x}$
- (12) $\cos(5x)$
- (13) $3^{\cos(5x)}$
- (14) $\cos(3^x)$
- (15) $\cos(\sqrt{x})$
- (16) $\sqrt{\cos x}$
- (17) $\ln(4x + 3)$
- (18) $\frac{1}{(\ln x)^4}$
- (19) $\tan(e^{-x} + 4x^2)$.
- (20) $\tan((2x + 1)^3 + x)$
- (21) $\tan^3 x$ (This means $(\tan x)^3$, so the “outer” function is x^3 , not $\tan x$.)
- (22) $\sin(x^3 - x)$
- (23) $((x^2 - x)^3 + x + 2)^4$
- (24) $\frac{1}{x^3 + 3^x}$.
- (25) $\cos(\ln(x^3 + 1))$
- (26) $\ln(\cos(x^3 + 1))$
- (27) $(\ln(\cos x))^3 + 1$
- (28) $e^{\sin(x^2)}$
- (29) $\sin(e^{x^2})$
- (30) $\sin^2(e^x)$. (This means $(\sin(e^x))^2$, so the “outer” function is x^2 , not $\sin x$.)

Answers.

- (1) $2(\sin x)(\cos x)$. (Let $u = \sin x$ to make the function u^2 .)
- (2) $(\cos(\sin x)) \cos x$. (Let $u = \sin x$ to make the function $\sin(u)$.)
- (3) $(\cos(\sin(\sin x))) \cos(\sin x) \cos x$. (Let $u = \sin(\sin x)$ to make the function $\sin(u)$.)
- (4) $(\cos(\sin(\sin(\sin x))))(\cos(\sin(\sin x))) \cos(\sin x) \cos x$. (Let $u = \sin(\sin(\sin x))$ to make the function $\sin(u)$.)
- (5) $\frac{1}{2\sqrt{x^2 + 5x}}(2x + 5)$. (Let $u = x^2 + 5x$ to make the function $\sqrt{u} = u^{1/2}$.)
- (6) $4(x^3 + x^2 + 1)^3(3x^2 + 2x)$. (Let $u = x^3 + x^2 + 1$ to make the function u^4 .)
- (7) $e^{-x^2/5}(-2x/5)$. (Let $u = -x^2/5$ to make the function e^u .)
- (8) $8(\sqrt{x} + x^3)^7 \left(\frac{1}{2\sqrt{x}} + 3x^2 \right)$. (Let $u = \sqrt{x} + x^3$ to make the function u^8 .)
- (9) $\frac{-1}{(\cos x)^2}(-\sin x) = \frac{\sin x}{\cos^2 x}$. (Let $u = \cos x$ to make the function $1/u$.)
- (10) $-\sin\left(\frac{1}{x}\right) \left(\frac{-1}{x^2}\right) = \frac{\sin(1/x)}{x^2}$. (Let $u = 1/x$ to make the function $\cos u$.)
- (11) $3^{\cos x}(\ln 3)(-\sin x) = -(\ln 3)3^{\cos x}(\sin x)$. (Let $u = \cos x$ to make the function 3^u .)
- (12) $-\sin(5x)5 = -5 \sin(5x)$. (Let $u = 5x$ to make the function $\cos u$.)
- (13) $3^{\cos(5x)}(\ln 3)(-\sin(5x))5 = -5(\ln 3)3^{\cos(5x)} \sin(5x)$. (Let $u = \cos(5x)$ to make the function 3^u .)
- (14) $-\sin(3^x)3^x(\ln 3) = -(\ln 3)(\sin(3^x))3^x$. (Let $u = 3^x$ to make the function $\cos u$.)
- (15) $-\sin(\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{-\sin(\sqrt{x})}{2\sqrt{x}}$. (Let $u = \sqrt{x}$ to make the function $\cos u$.)
- (16) $\frac{1}{2\sqrt{\cos x}}(-\sin x) = \frac{-\sin x}{2\sqrt{\cos x}}$. (Let $u = \cos x$ to make the function \sqrt{u} .)
- (17) $\frac{1}{4x + 3}(4) = \frac{4}{4x + 3}$. (Let $u = 4x + 3$ to make the function $\ln u$.)
- (18) $-\frac{1}{(\ln x)^8}(4(\ln x)^3)\frac{1}{x} = \frac{-4(\ln x)^3}{x(\ln x)^8}$. (Let $u = (\ln x)^4$ to make the function $1/u$.)
- (19) $(\sec^2(e^{-x} + 4x^2))(-e^{-x} + 8x)$. (Let $u = e^{-x} + 4x^2$ to make the function $\tan u$.)
- (20) $(\sec^2((2x + 1)^3 + x))(3(2x + 1)^2 + 1)(2) = 2(3(2x + 1)^2 + 1) \sec^2((2x + 1)^3 + x)$. (Let $u = (2x + 1)^3 + x$ to make the function $\tan u$.)
- (21) $3 \tan^2 x \sec^2 x$. (Let $u = \tan x$ to make the function u^3 .)
- (22) $(\cos(x^3 - x))(3x^2 - 1)$. (Let $u = x^3 - x$ to make the function $\sin u$.)
- (23) $4((x^2 - x)^3 + x + 2)^3(3(x^2 - x)^2(2x - 1) + 1)$. (Let $u = (x^2 - x)^3 + x + 2$ to make the function u^4 .)
- (24) $\frac{-1}{(x^3 + 3x)^2}(3x^2 + 3^x \ln 3)$. (Let $u = x^3 + 3^x$ to make the function $1/u$.)

$$(25) -\sin(\ln(x^3 + 1)) \frac{1}{x^3 + 1} (3x^2) = \frac{-3x^2 \sin(\ln(x^3 + 1))}{x^3 + 1}. \text{ (Let } u = \ln(x^3 + 1) \text{ to make the function } \cos u.)$$

$$(26) \frac{1}{\cos(x^3 + 1)} (-\sin(x^3 + 1))(3x^2) = \frac{-3x^2 \sin(x^3 + 1)}{\cos(x^3 + 1)}. \text{ (Let } u = \cos(x^3 + 1) \text{ to make the function } \ln u.)$$

$$(27) 3(\ln(\cos x))^2 \frac{1}{\cos x} (-\sin x) = \frac{-3(\ln(\cos x))^2 \sin x}{\cos x}. \text{ (Let } u = \ln(\cos x) \text{ to make the function } u^3 + 1.)$$

$$(28) e^{\sin(x^2)} (\cos(x^2))(2x) = 2x \cos(x^2) e^{\sin(x^2)}. \text{ (Let } u = \sin(x^2) \text{ to make the function } e^u.)$$

$$(29) (\cos(e^{x^2})) e^{x^2} (2x) = 2x e^{x^2} \cos(e^{x^2}). \text{ (Let } u = e^{x^2} \text{ to make the function } \sin u.)$$

$$(30) 2(\sin(e^x)) (\cos(e^x)) e^x = 2e^x \sin(e^x) \cos(e^x). \text{ (Let } u = \sin(e^x) \text{ to make the function } u^2.)$$