

Math 1131 Applications: Continuity

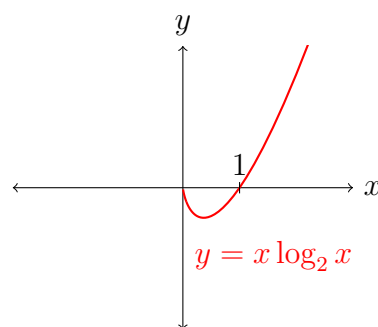
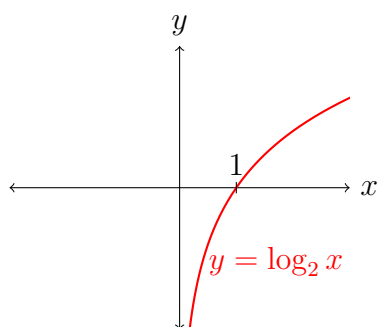
Continuity of a function, which means $\lim_{x \rightarrow a} f(x) = f(a)$ for each a in the domain of $f(x)$, tells us that *nearby inputs in the domain lead to nearby outputs*. This is a feature of most functions you know about: polynomials, power functions (like \sqrt{x}), exponential and logarithmic functions, $\sin x$, and $\cos x$ are all continuous. The notion of continuity gives a *name* to this property. Why does the property matter?

- The whole idea of approximate calculations depends on continuity. To estimate $\log_{10} \pi$, you can't plug π exactly into a calculator or computer since they store numbers to a limited amount of decimals, maybe 20 digits, but π is not a 20-digit decimal: its decimal expansion never ends. Since $\log_{10} x$ is continuous, you can *estimate* $\log_{10} \pi$ by using a decimal approximation of π in place of π .
- Equations for how a physical system evolves have parameters, like the initial position and velocity of an object used to predict its future position. In practice parameters (or any measurements) are known only approximately, so we want solutions to the equations to have *continuous dependence on the parameters*.

Example. Solutions of $x^2 - tx + 1 = 0$ are $x = (t + \sqrt{t^2 - 4})/2$ and $x = (t - \sqrt{t^2 - 4})/2$, which are continuous in t (if $|t| \geq 2$, so the solutions make sense in \mathbf{R}). This says solutions of $x^2 - tx + 1 = 0$ depend continuously on t .

Remark. Another feature of interest besides continuous dependence on parameters is sensitive dependence on initial conditions, also known as the [butterfly effect](#). It is studied in [chaos theory](#).

- Continuity can suggest how to define a function at a point where it is not initially defined. Consider the functions $f(x) = \log_2 x$ and $g(x) = x \log_2 x$ when $x > 0$, shown below. Can we extend the functions to $x = 0$?

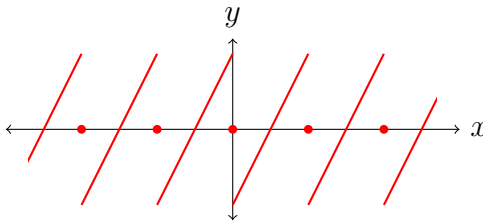


Since $f(x)$ and $g(x)$ are continuous for $x > 0$, the most natural way to extend them to $x = 0$ is by limits: $f(0) := \lim_{x \rightarrow 0^+} \log_2 x$ and $g(0) = \lim_{x \rightarrow 0^+} x \log_2 x$. As the graphs above suggest, $\lim_{x \rightarrow 0^+} \log_2 x = -\infty$ and $\lim_{x \rightarrow 0^+} x \log_2 x = 0$. (We will explain the second limit later in the course.) Thus it's natural to set $\log_2 0 = -\infty$ and $0 \log_2 0 = 0$ by “continuity from the right”.

Continuity is the most basic “nice” property a function can have. There are nicer properties we can hope for, with labels like differentiability, smoothness, and analyticity; you won't see the last two in this course.

We have emphasized the importance of continuity, but discontinuities (or approximations to them) do show up in the real world. Here are examples of jump discontinuities.

1. A [sawtooth wave](#), shown below, is a fundamental example on synthesizers and in signal processing. To hear a sawtooth wave, see the video [here](#).



2. Electronics on the F-22 (picture below on left) [shut off](#) in 2007 when crossing the International Date Line, where longitude jumps from -180° to 180° .



3. At the edge of a shock wave (picture above on right), pressure nearly has a jump discontinuity.

4. A [phase transition](#) of matter is often accompanied by a jump discontinuity in a derivative of a thermodynamic parameter (*e.g.*, density or Gibbs free energy).

5. The energy levels of an electron are discrete, not continuous. Many physical quantities (energy, charge, light) that at ordinary scales appear to vary continuously at the atomic scale are discrete (physicists say “quantized”).

6. The media you use present the illusion of continuity but are ultimately discrete (pixels on a computer screen, 24 frames per second in a movie) . An amusing example where the continuity illusion is broken is the helicopter blades [here](#). Also check out the rolling shutter effect [here](#).