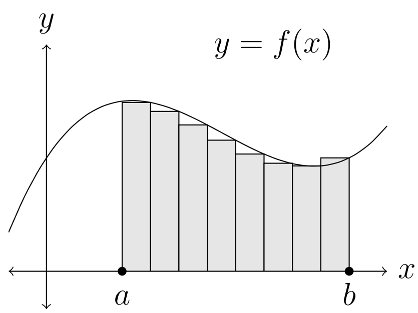
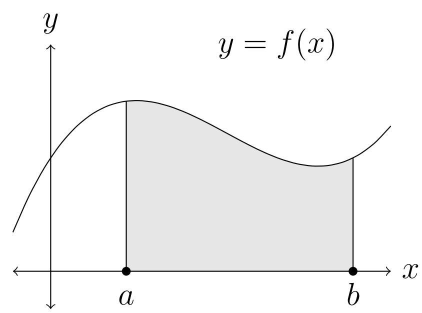
Math 1131 Applications: Definite Integrals

In our course, a definite integral is described as a limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x,$$

where $\Delta x = (b-a)/n$ and x_k^* is a point in the kth subinterval after $[a, b]$ is broken up into n subintervals. When $f(x) > 0$ this definite integral is the area under the graph of $y = f(x)$ between $x = a$ and $x = b$.





Many physical problems give rise to definite integrals, often for reasons that are not directly about areas. Integrals occur when you're *adding up many small quantities*. Here are several such examples.

1. A PID controller, which stands for proportional-derivative-integral controller, uses approximations to integrals and derivatives of error terms to create an output that is a preferred value. It is used in automatic steering systems in ships and cars and in the propellor control of quadcopters. Overshooting the desired height of a quadcopter results in a negative contribution to an integral term in its controller, which is a physical example of integrating negative values.

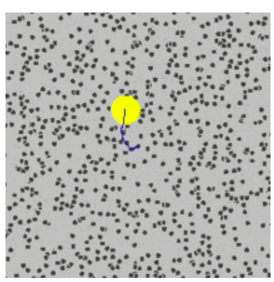
2. The definite integrals $\int_0^x \sin(t^2)\,dt$ and $\int_0^x \cos(t^2)\,dt$ with variable x are called Fresnel integrals and they are used in highway or railroad track design to create transition curves where a road or track changes from a straight path to a curved path and engineers want to avoid uncomfortable forces during the transition.

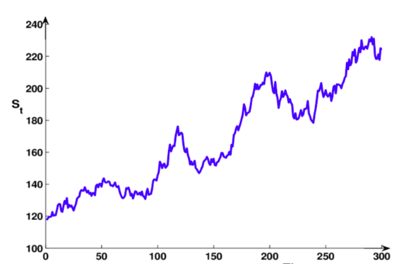
3. The center of mass of a body is determined by integrals (usually multivariable integrals) involving the body's density function. Its location is important for determining the stability of a moving body like a car, airplane, boat, and rocket. For example, the Swedish warship Vasa was built in the 1620s when methods of determining ship stability were still rather crude. It was too top-heavy and tipped over on its first day at sea.

4. Potential energy arises in many forms: gravitational potential energy, electrical potential energy (related to voltage), chemical potential energy, and so on. What is important is not potential energy values directly, but potential energy *differences*, and these are expressed by the Fundamental Theorem of Calculus as the definite integral of a suitable potential function between two points.

5. The speed a rocket needs to escape Earth's gravitational pull is called its escape velocity, and its value can be obtained by calculating a definite integral.

6. The random movement of a particle in a fluid from bombardment by much smaller molecules and the random up and down movement of stock prices are both modeled by a process called Brownian motion (see pictures below).





Central to the mathematical description of Brownian motion is definite integrals such as $\int_0^x \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}\,dy$, which describe a family of probability bell curves ("normal distributions") as t and x vary.

7. There are many ways to turn one function into another using integration with a new parameter. These are called integral transforms. Three named examples are the Fourier transform used in signal processing and crystallography, the Laplace transform used in control theory and to solve differential equations, and the Radon transform used in medical imaging. The 1979 Nobel prize in medicine was awarded in part to Allan Cormack for his mathematical work involving the Radon transform. His Nobel prize speech can be watched here and he mentions integrals a few times between 5:00 and 7:30.