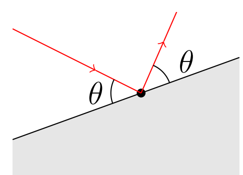
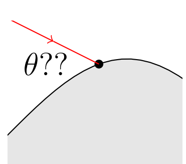
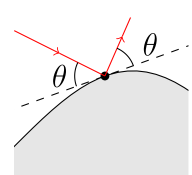
Math 1131 Applications: Tangents

Background. In physics there is the widely taught principle “the angle of incidence equals the angle of reflection” for a light ray (or sound wave, or other propagating signal) bouncing off a flat surface, as in the picture below. The angle θ which the light ray has with the surface is the same angle θ at which it is reflected off the surface.



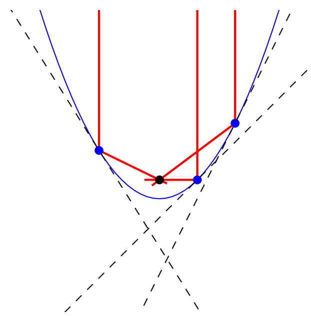
We can make light rays hit a *curved* surface, but how do we express “the angle of incidence equals the angle of reflection” in that case? What is the angle that a light ray makes at the point where it meets the surface? (See the picture below on the left.) The point where two lines meet has an angle, but at first it may not be clear how to define the angle where a line meets a curve. The answer is to use the angle θ between the line and the *tangent line* to the curve at the point of incidence, as in the picture below on the right, where the tangent line to the curve at the marked point is the dashed line. The same tangent line is used to measure the angle of reflection for the light ray. Notice the similarity between the tangent line below and the flat wall in the first picture above: at the point of contact, the curved surface reflects the light ray exactly like the flat wall.





The lesson here is that the angle of incidence equals the angle of reflection for light rays (or other types of signals) that meet a curved surface when we use the *tangent line* to a curve at a point as a flat (that is, linear) approximation to the curve at that point. The mathematical tool that lets us compute tangent lines to curves in a systematic way is calculus.

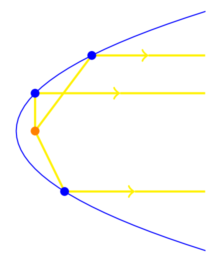
Application 1. (Parabolic tangents) Parabolas have the following important geometric property: when we orient the parabola upright, vertical lines that come down onto the parabola and reflect off the parabola with the angle of incidence equal to the angle of reflection (the angle is measured using tangent lines to the parabola at the incident points on it) all pass through a common point. The common meeting point of the reflected light rays is called the *focus* of the parabola and it can be computed from an equation of the parabola. For example, the vertical red light rays in the picture below all reflect at a tangent line (the dashed lines) and the reflections all pass through the common black point.



This geometric property is true for all parabolas, not just upright parabolas. Every parabola, no matter how it is oriented, has a line of symmetry down the middle, and the geometric property we described holds for all light rays (or other incoming signals) that are parallel to the symmetry line: if they reflect off the parabola with the angle of incidence equal to the angle of reflection then the reflected light rays all pass through the focus of the parabola, which lies on the parabola's line of symmetry.

That all incoming light rays parallel to a parabola's line of symmetry reflect off the parabola and pass through a common point is why deep space radars are shaped like parabolic surfaces with the receiver at the focus. This idea is not limited to light ray signals from distant stars. It is also why TV satellite dishes are shaped like part of a parabolic surface with the feed horn of the satellite dish at the focus.

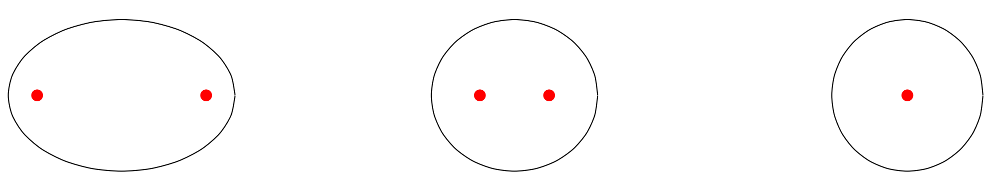
Exchanging incoming signals with outgoing signals, part of a reflective parabolic surface with a light source placed at the focus (orange point in picture below) sends out parallel light beams. This is is why car headlights are parabolic: parallel outgoing beams make the road ahead more visible. It is *not* desirable for outgoing light beams on rear car lights to be parallel (why?), so they are not enclosed in parabolic reflectors.



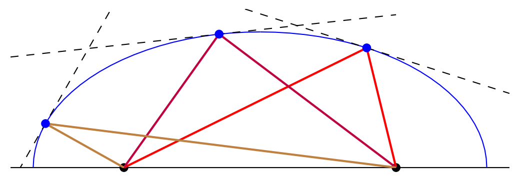
The reflective properties of parabolas lead to an optical illusion shown in the following two videos:

- Zach Star here (0:00-0:30 and 9:35-11:30),
- Mathologer here (0:00-0:45, 5:30-6:36, and 9:44-10:35, with more mathematical background in 1:35-5:30.)

Application 2. (Elliptic tangents) An ellipse is a stretched circle, as in the pictures below. It is the points in a plane whose distances to two specified points, called the foci (marked in red in the pictures), add up to a constant value. A circle (third picture) is an ellipse whose foci coincide at the center of the circle.



If the walls of a room are shaped like part of an ellipsoid (a stretched sphere, so a 3-d version of an ellipse) then sound waves starting at one focus reflect off the wall (with angle of incidence equal to angle of reflection) and reach the other focus along paths of *equal length*, hence equal time since sound travels at a common speed within a room. Some paths from one focus to another are illustrated in the diagram below.



That a sound made at one focus reaches the other focus at the same time from every reflection off the wall (always following the rule that the angle of incidence equals the angle of reflection, which is how tangent lines are involved) means the sound is heard clearly at the other focus instead of being muffled or washed out like at other points distant from the first focus, where reflections of the same sound arrive at different times. This is described in the same Zach Star video mentioned before (7:07-7:40). A similar effect can be achieved with other shapes, like two parabolic reflectors, which is shown in the Mathologer video mentioned before (6:36-9:44). Find other videos illustrating this acoustic effect by googling “whispering gallery”.

A medical application of the reflective property of ellipses is lithotripsy, a noninvasive method of treating kidney stones. It uses a machine, called a lithotripter, with a half-elliptic cavity. Shock waves generated at one focus, outside the body, reflect off the cavity and pulverize the kidney stone when it is placed at the other focus,

In the Numberphile video “Elliptical Pool Table” here (screenshots below), on a pool table with an elliptical boundary a pool shot that passes through one focus will sink into a hole at the other focus. (The shot need not start at that focus, which is an error in the video at 1:17-1:23.) It is fun to watch! The role of angles at the boundary measured with a tangent line is shown in 0:57-1:14.