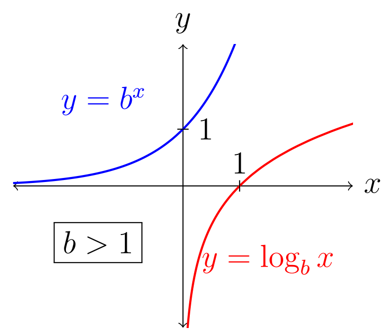
Math 1131 Applications: Logarithms

Background. Logarithms are inverse to exponentials: $y = \log_b x$ is the same as $x = b^y$. The standard case is $b > 1$, which is shown in the graph below. Know it well.



The following algebraic rules show how logarithms are inverse to exponentials.

$b^{\log_b x} = x$	$\log_b(b^x) = x$
$b^u b^v = b^{u+v}$	$\log_b(xy) = \log_b x + \log_b y$
$\dfrac{b^u}{b^v} = b^{u-v}$	$\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
$(b^u)^v = b^{uv}$	$\log_b(x^y) = y \log_b x$

There are two features of logarithms (to a base greater than 1) that are worth remembering to get an intuition for them:

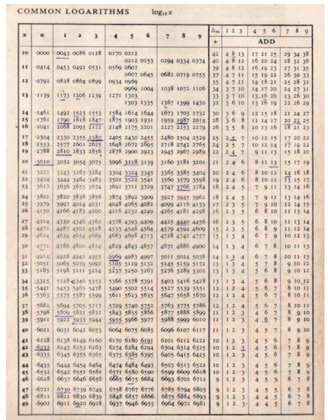
- They grow at a slower and slower rate (see graph above). For example, the intervals $[1, 10000]$ and $[50000, 60000]$ are equally long, but $[\log_2 1, \log_2 10000] \approx [0, 13.2]$ is short and $[\log_2 50000, \log_2 60000] \approx [15.6, 15.8]$ is even shorter.

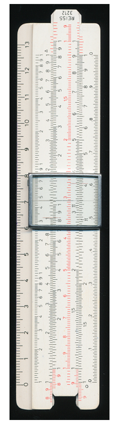
- The change of base formula lets us convert from logarithms in one base b to logarithms in another base c:

$$\log_c x = \frac{\log_b x}{\log_b c} = \frac{1}{\log_b c} \log_b x.$$

  For example, $\log_2 x = \dfrac{\log_{10} x}{\log_{10} 2}$: logarithms base 10 and base 2 are the same up to scaling by $1/\log_{10} 2 \approx 3.32$. (This is like measuring length: feet and meters are the same up to a conversion factor.) One way of expressing this is that up to an overall scaling factor, there is essentially only one logarithm function!

Applications. That $\log_b(xy) = \log_b x + \log_b y$ means logarithms turn multiplication into addition, which is simpler. This is why, for centuries, logarithm tables or slide rules (see below) were used in navigation, astronomy and engineering.





Calculators and computers made logarithm tables and slide rules obsolete, but they did not make logarithms as a mathematical tool obsolete! For example, some common scientific measuring scales are based on logarithms:

- the Richter scale to measure the intensity of earthquakes,
- the pH scale to measure the acidity of a solution in water,
- the decibel scale to measure the intensity of sound.

The way we perceive stimulus changes appears to be logarithmic: look up the Ferry–Porter law, Fitts's Law, and the Weber–Fechner law (a Numberphile video on the topic is here). The idea is that our perception P of a stimulus change depends linearly on the *logarithm* of the intensity I of the stimulus: $P = k\log(I) + \ell$. (Use any base for the logarithm: changing the base just changes k by the change of base formula, *e.g.*, $k\log_2 x + \ell = (k/\log_{10} 2)\log_{10} x + \ell$.)

Entropy is based on logarithms. It was first used in thermodynamics and later in information theory, which is applied in data compression and signal/image processing.

In finance, the Rule of 72 for the doubling time of an investment is based on both a logarithm calculation and on 72 being divisible by many small numbers.

In many naturally occurring data sets, leading digits are often not equally likely but instead are distributed in a logarithmic way described by Benford's law. This was discovered in the late 1800s and early 1900s by scientists using logarithm tables. It is used nowadays by accountants and lawyers for financial fraud detection.