MATH 1131Q Final Exam Practice Problems - Solutions

1 – Be sure to review Exams 1 and 2 and their practice sets, as well as other materials like worksheets and quizzes!

2 - A certain function $f(x)$ satisfies $f''(x) = 2 - 3x$ with $f'(0) = -1$ and $f(0) = 1$. Compute $f(2)$.

$f'(x) = 2x - \frac{3}{2}x^2 + C$, $f'(0) = C = -1 \rightarrow f'(x) = 2x - \frac{3}{2}x^2 - 1$

So $f(x) = x^2 - \frac{1}{2}x^3 - x + D$, $f(0) = D = 1 \rightarrow f(x) = x^2 - \frac{1}{2}x^3 - x + 1$

$\rightarrow f(2) = 4 - 4 - 2 + 1 = \underline{-1}$

3. Find $f(x)$ if $f'(x) = 3x^2 + \dfrac{2}{x}$ for $x > 0$ and $f(1) = 3$.

$f'(x) = 3x^2 + 2 \cdot \frac{1}{x}$

$\rightarrow f(x) = x^3 + 2 \cdot \ln|x| + C$

$f(1) = 3$

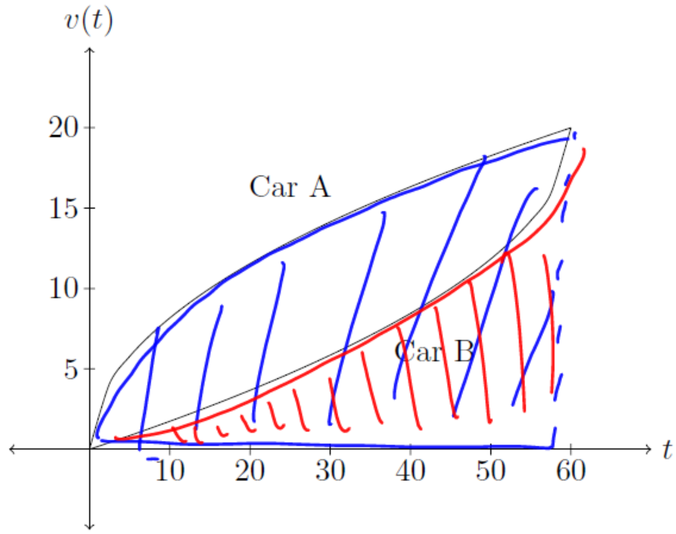
$\rightarrow f(1) = 1^3 + 2\ln(1) + C = 3$

$1 + 2(0) + C = 3$

$C = 2$

$\rightarrow \boxed{f(x) = x^3 + 2\ln x + 2}$

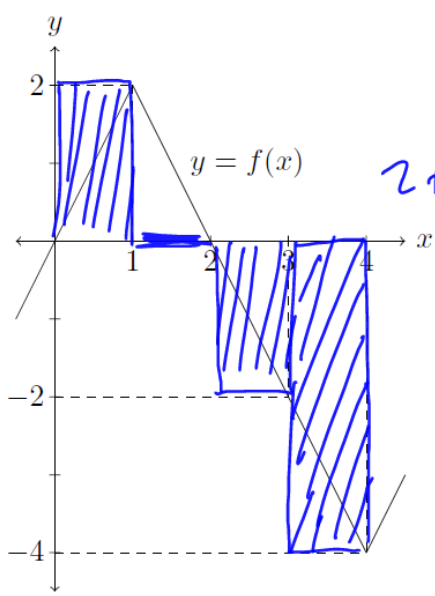
4 Below is the graph of the velocity (measured in ft/sec) over the interval $0 \leq t \leq 60$ for two cars, Car A and Car B. How do the distances traveled by each compare at over this interval?



Car A has traveled further than Car B (blue area bigger than red)

5 If we use a right endpoint approximation with four subintervals (i.e., R_4), then what is the resulting approximation for

$$\int_0^4 f(x)\,dx?$$



$2+0-2-4=-4$

6 Evaluate the definite integral $\int_{-1}^{1} (x^2 + 2x + 1)\, dx.$ $= \left[\frac{x^3}{3} + x^2 + x\right]_{-1}^{1}$

$= \left(\frac{1}{3} + 1 + 1\right) - \left(-\frac{1}{3} + 1 - 1\right)$

$= \frac{2}{3} + 2 = \frac{8}{3}$

7 Assume that $\int_{-2}^{3} f(x)\, dx = 4$. What is the value of $\int_{-2}^{3} (f(x) + 1)\, dx$?

(A) 4 (B) 5 (C) 6

(D) 9 (E) 20

$= \int_{-2}^{3} f(x)dx + \int_{-2}^{3} 1\, dx$

$= 4 + 1(3 - (-2))$

$= 4 + 5 = 9$

8 Which of the following is the derivative of the function

$$f(x) = \int_{1}^{x^2} \frac{1}{t^3 + 1}\, dt?$$

$f(x) = g(u)$ with $u(x) = x^2$ and $g(x) = \int_{1}^{x} \frac{1}{t^3+1}\, dt.$

(A) $\frac{2x}{x^6 + 1}$ (B) $\frac{1}{x^6 + 1}$ (C) $\frac{2x}{x^5 + 1}$

(D) $\frac{1}{x^3 + 1}$ (E) $\frac{2x}{x^3 + 1}$

So $f'(x) = g'(u)\frac{du}{dx}$

$= \frac{1}{u^3+1} \cdot 2x$

$= \frac{2x}{(x^2)^3+1} = \frac{2x}{x^6+1}$

9 $w'(t) = \frac{\ln(t)}{t}$

$$\int_5^{10} w'(t)\,dt = \int_5^{10} \frac{\ln t}{t}\,dt \quad , \quad \text{let } u = \ln t$$

$$dt = \frac{1}{t}\,dt$$

$$t=5, \ u = \ln 5$$

$$t = 10 \ u = \ln 10$$

$$= \int_{\ln 5}^{\ln 10} u\,du$$

$$= \frac{u^2}{2}\Bigg|_{\ln 5}^{\ln 10} = \frac{(\ln(10))^2}{2} - \frac{(\ln(5))^2}{2}$$

$$\approx 1.36 \text{ pounds.}$$

The integral of a rate of change gives net change So this means the child gained about 1.4 pounds between ages 5 and 10.

10 a) $$\int_0^{\pi/4} \frac{1+\cos^2 x}{\cos^2 x}\,dx = \int_0^{\pi/4} \frac{1}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}\,dx$$

$$= \int_0^{\pi/4} \sec^2 x + 1\,dx$$

$$= \tan x + x \Big|_0^{\pi/4}$$

$$= \left(\tan(\pi/4) + \pi/4\right) - \left(\tan(0) + 0\right)$$

$$= \boxed{1 + \pi/4}$$

b) $\int_0^1 x^{10} + 10^x \, dx = \left. \frac{x^{11}}{11} + \frac{10^x}{\ln(10)} \right|_0^1$

$= \left(\frac{1}{11} + \frac{10}{\ln 10}\right) - \left(0 + \frac{1}{\ln(10)}\right)$

$= \boxed{\frac{1}{11} + \frac{9}{\ln 10}}$

c) $\int \left(\frac{1+r}{r}\right)^2 dr = \int \frac{1+2r+r^2}{r^2} \, dr$

$= \int \frac{1}{r^2} + \frac{2}{r} + 1 \, dr$

$= \boxed{-\frac{1}{r} + 2\ln|r| + r + C}$

d) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$ let $u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2} x^{-1/2} \, dx$

$du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx$

$= \int e^u \cdot 2 \, du = 2e^u + C$

$= \boxed{2e^{\sqrt{x}} + C}$

e) $\int_5^{10} \frac{dt}{(t-4)^2}$ let $u = t-4$, $du = dt$ $t=5 \Rightarrow u=1$, $t=10 \Rightarrow u=6$

$$\int_1^6 \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_1^6 = \frac{-1}{6} - \left(-\frac{1}{1}\right) = \boxed{5/6}$$

(*) f) $\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$ let $u = e^x$ $du = e^x dx$

Challenge question

Note: $u^2 = e^{2x}$ $x=0$ $u=e^0=1$; $x=1$ $u=e$

$$\int_1^e \frac{1}{1+u^2}\,du$$

Recall: $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$$= \arctan u\Big|_1^e = \arctan(e) - \arctan(1)$$
$$= \boxed{\arctan(e) - \pi/4}$$

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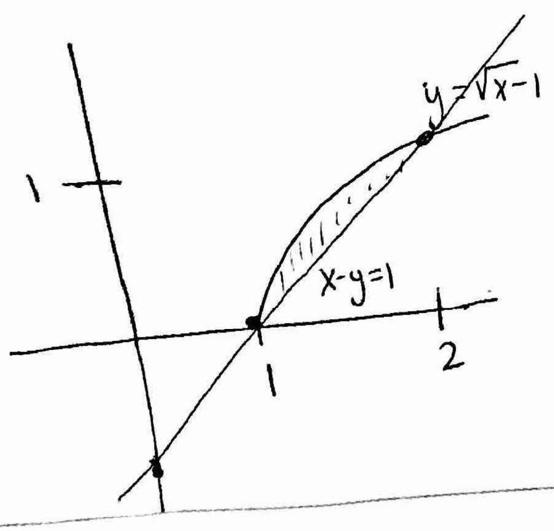

$x-y=1 \Rightarrow y = x-1$
& $y = \sqrt{x-1}$

find intersections:

$x-1 = \sqrt{x-1}$

Square both sides $x^2-2x+1 = x-1$

$x^2-3x+2 = 0$

$(x-2)(x-1) = 0$

$x = 1, 2$

Area: $\int_1^2 \sqrt{x-1} - (x-1)\,dx$ $u = x-1$, $du = dx$ $x=1, u=0$; $x=2, u=1$

$$= \int_0^1 u^{1/2} - u\,du = \frac{2}{3}u^{3/2} - \frac{u^2}{2}\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$$

$-\left(-\frac{1}{4}+\frac{5}{2}-\frac{1}{6}\right)$

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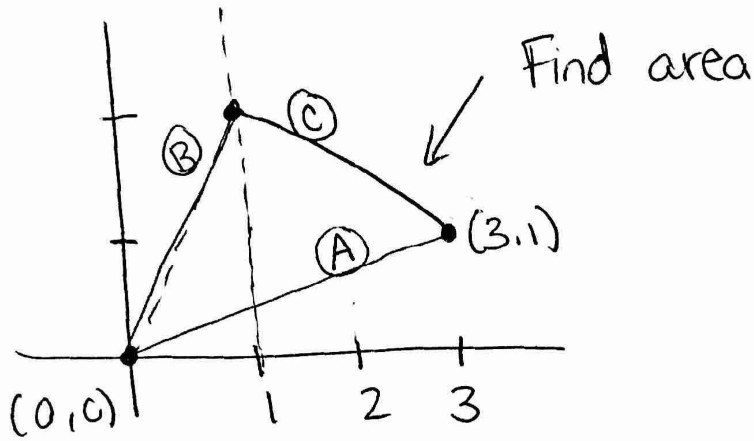


Area: $\int_0^1 Ⓑ - Ⓐ \, dx + \int_1^3 Ⓒ - Ⓐ \, dx$

Ⓐ line through $(0,0)$ and $(3,1)$

Slope: $\frac{1}{3}$ point: $(0,0)$

$y = \frac{1}{3}x$

Ⓑ line through $(0,0)$ and $(1,2)$

Slope: 2 point $(0,0)$

$y = 2x$

Ⓒ line through $(1,2)$ & $(3,1)$

Slope $\frac{2-1}{1-3} = \frac{1}{-2} = -\frac{1}{2}$ point: $(1,2)$

$y - 2 = -\frac{1}{2}(x-1) \rightarrow y = -\frac{1}{2}x + \frac{1}{2} + 2$

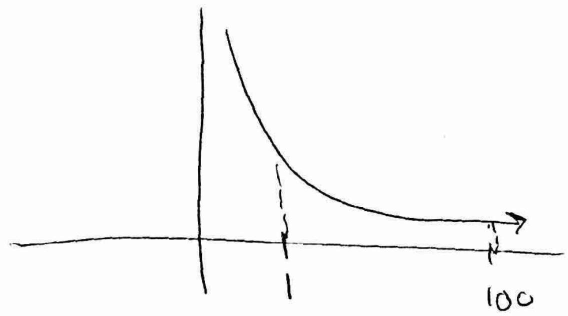
$= -\frac{1}{2}x + \frac{5}{2}$

Area:

$$\int_0^1 2x - \frac{1}{3}x \, dx + \int_1^3 \left(-\frac{1}{2}x + \frac{5}{2}\right) - \frac{1}{3}x \, dx = \boxed{\frac{5}{2}}$$

$$= x^2 - \frac{1}{6}x^2 \Big|_0^1 + -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{1}{6}x^2 \Big|_1^3 = \left(1 - \frac{1}{6}\right) + \left(-\frac{9}{4} + \frac{15}{2} - \frac{9}{6}\right) - \left(-\frac{1}{4} + \frac{5}{2} - \frac{1}{6}\right)$$

13 $y = 1/x$



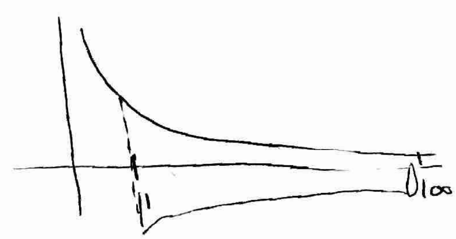
a) $\displaystyle\int_1^{100} \frac{1}{x}\,dx = \ln x \Big|_1^{100} = \ln(100) - \ln(1)$

$= \boxed{\ln(100)}$

b) as $a \to \infty$ $\displaystyle\int_1^{a} \frac{1}{x}\,dx = \ln x \Big|_1^{a} = \ln(a)$

as $a \to \infty$ $\ln(a) \to \infty$, so we get infinite area.

c) Volume:



$\displaystyle\pi \int_1^{100} \left(\frac{1}{x}\right)^2 dx = \pi \int_1^{100} \frac{1}{x^2}\,dx$

$\displaystyle= \pi \left(-\frac{1}{x} \Big|_1^{100} \right)$

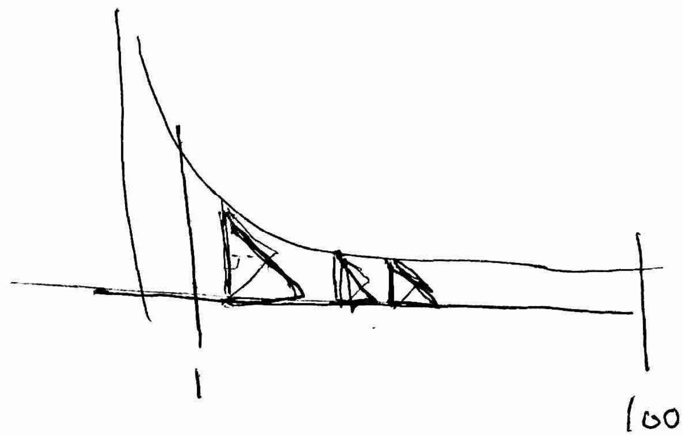
$\displaystyle= \pi \left(-\frac{1}{100} + 1 \right) = \frac{99}{100}\pi$

d) as $a \to \infty$ $\displaystyle\pi \int_1^{a} \left(\frac{1}{x}\right)^2 dx = \pi \left(-\frac{1}{x} \Big|_1^{a} \right) = \pi \left(1 - \frac{1}{a}\right)$

So as $a \to \infty$ Volume goes to: $\displaystyle\lim_{a \to \infty} \pi\left(1 - \frac{1}{a}\right) = \boxed{\pi}$

↳ Volume is finite even though area is infinite!! Strange but true!! More to come in calc 2!! :)

e) Volume w/ cross sections $\perp$ to x-axis right triangles whose height is half their base.



$$\int_1^{100} \begin{pmatrix}\text{Area of}\\ \text{Slice}\end{pmatrix} dx$$

$$= \int_1^{100} \frac{1}{2}(\text{base})(\text{height})\, dx$$

base is under $1/x$ $\rightarrow$ $1/x$

height is $\frac{1}{2}$ b = $1/2x$

$$= \int_1^{100} \frac{1}{2}\left(\frac{1}{x}\right)\left(\frac{1}{2x}\right) dx$$

$$= \int_1^{100} \frac{1}{4x^2}\, dx = \frac{-1}{4x}\Big|_1^{100} = \frac{-1}{400} + 1/4$$

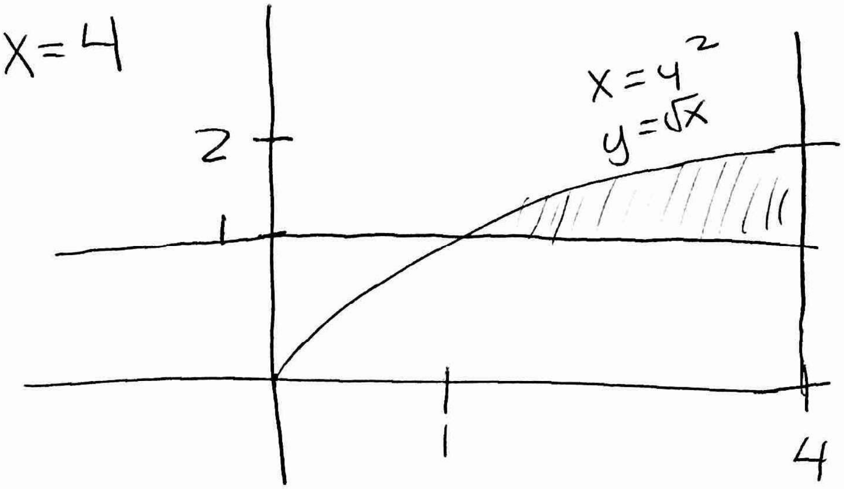
$$= 1/4 - \frac{1}{400} = \boxed{\frac{99}{400}}$$

f) as $a \to \infty$ $\int_1^a \frac{1}{4x^2} dx = \frac{-1}{4x}\Big|_1^a$

$$= 1/4 - \frac{1}{4a}$$

$$\lim_{a \to \infty} 1/4 - \frac{1}{4a} = \boxed{1/4}.$$ (again finite volume!)

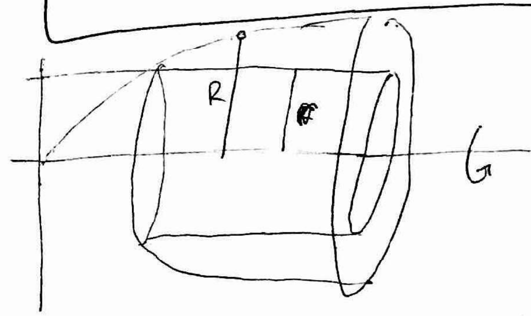
14 $y=\sqrt{x}$, $y=1$, $x=4$



a) Area of region

$$\boxed{\int_1^4 \sqrt{x}-1 \; dx} \quad \boxed{\text{or}} \quad \boxed{\int_1^2 4-y^2 \; dy}$$

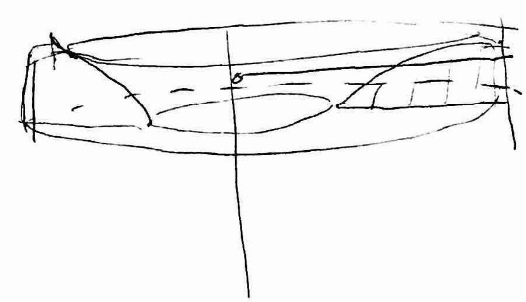
b) $\pi\int \underset{\text{outer radius}}{R}^2 - \underset{\text{inner radius}}{r}^2 \; dx$



$$= \pi\int_1^4 (\sqrt{x})^2 - (1)^2 \, dx$$

$$= \boxed{\pi\int_1^4 x-1 \; dx}$$

Rotate about x-axis



c) Rotate about y-axis

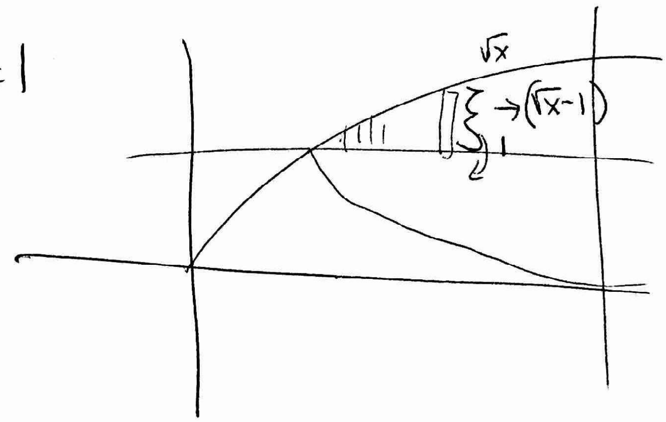
$$\pi\int_1^2 \left(\begin{smallmatrix}\text{outer}\\ \text{Radius}\end{smallmatrix}\right)^2 - \left(\begin{smallmatrix}\text{inner}\\ \text{radius}\end{smallmatrix}\right)^2 dy$$

$$= \pi\int_1^2 4^2 - (y^2)^2 \, dy$$

$$= \boxed{\pi\int_1^2 16-y^4 \; dy}$$

d) Rotate around $y=1$

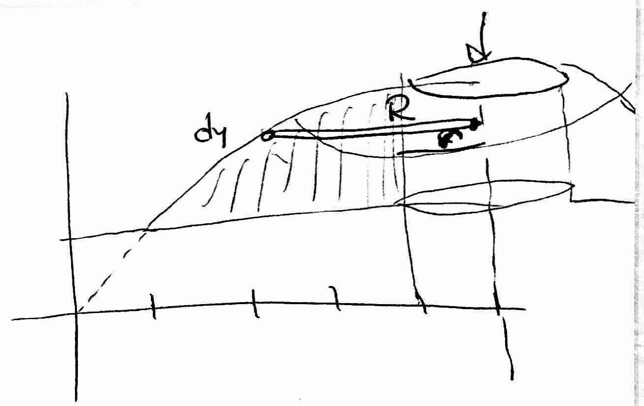
$$\pi \int_1^4 R^2 \, dx$$



$$\boxed{\pi \int_1^4 (\sqrt{x}-1)^2 \, dx}$$

or $$\boxed{\pi \int_1^4 x - 2\sqrt{x} + 1 \, dx}$$

e) Rotate aroung $x=5$



$$\pi \int_1^2 R^2 - r^2 \, dy$$

R: outer radius from $x=y^2$ to $x=5$

subtract right − left $(5-y^2)$

r: inner radius from $x=4$ to $x=5$

Subtract $5-4 = \boxed{1}$

$$\pi \int_1^2 (5-y^2)^2 - 1^2 \, dy$$

$$= \boxed{\pi \int_1^2 (5-y^2)^2 - 1 \, dy}$$