Triple Integrals in Cartesian Coordinates

In this demo, we will look at visualizing the domain of a triple integral and changing the order of integration. The triple integral we are focusing on is

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dzdydx$$

First, let's visualize the domain of integration. Reading inside to out, we see that z goes from $z = 0$ to $z = 1 - y$, then in the xy plane, y goes from $\sqrt{x}$ to 1 and x goes from 0 to 1. Let's graph the important boundary surfaces.

1. Go here `https://www.geogebra.org/3d/pykdnuga`.

2. Turn on the first three entries. This will plot a piece of the surface $y = x^2$, the plane $z = 1 - y$ and the plane $x = 0$. See if you can find the region where z is below $1 - y$ and y goes from $sqrtx$ to 1.

3. It's sometimes easier to picture the region is we can visualize the curves of intersection between the surfaces. Turn on the next 6 curves (b-h). You might want to Toggle the first 3 surfaces on and off to really picture the region.

4. Go back and turn off the first three surfaces so you just have the six outline curves. Then turn on the next three surfaces to see just the parts of the surfaces that or on the boundary of what you are integrating over(i-k).

5. Now that we have this region, we want to decide how many integral we need in each of the different setups, i.e. can we set this up as on integral if we do x first, y first and z first. To determine this, we want to see if when we move in the direction of a specific axis, if we always enter and exit the region along the same surfaces.

6. Let's look at the x-first integral (Type x) first. Scroll down and turn on the "xToothpick." This produces a line in the direction of the x axis. Move around y_0 and z_0 above it until the line pierces the surfaces. When it pierces it, does it always pierce that same surfaces? (We should see in we move along the line from back to front, we first pierce light blue, then dark blue.)

7. Now, turn off the "xToothpick" and turn on the "yToothpick." What surfaces do we pierce? What about for the "zToothpick?"

8. We should see that this surface is Type x, Type y and Type z since we always pierce the same two surfaces regardless of the values of the other variables.

9. Now, we can also picture the projections into the coordinate planes by moving the axes around. See if you can find the projections into each of the three coordinate planes.