

1. The distance traveled by a particle in t seconds is given by $s(t) = t^2 + 3t$. What is the particle's average velocity over the interval $1 \leq t \leq 4$? [1]

- (A) 8 (B) 0 (C) 2
(D) 5 (E) -1

$$\begin{aligned} \frac{s(4) - s(1)}{4 - 1} &= \frac{(4^2 + 3 \cdot 4) - (1^2 + 3 \cdot 1)}{3} \\ &= \frac{16 + 12 - 1 - 3}{3} \\ &= \frac{24}{3} = \underline{\underline{8}} \end{aligned}$$

2. Evaluate the following limit: [1]

$$\lim_{x \rightarrow 1^-} \frac{x-3}{x-1}$$

- (A) 2 (B) -2 (C) -1
(D) $+\infty$ (E) $-\infty$

$$x = 0.999: \quad \frac{x-3}{x-1} = \frac{-2.001}{-0.001} = 2001$$

large,
positive

$\rightarrow \underline{\underline{\infty}}$

3. Using the table below, what appears to be the value of the limit

[1]

$$\lim_{x \rightarrow 2^+} f(x)$$

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	3	7	291	4081	?	-9532	-112	-17	-1

- (A) ∞ (B) $-\infty$ (C) 0
 (D) -1000 (E) None of the above.

looks like $\lim_{x \rightarrow 2^+} f(x) = -\infty$

4. If $\lim_{x \rightarrow 3^+} f(x) = 5$ what can be said about $\lim_{x \rightarrow 3^-} f(x)$?

[1]

- (A) It must be 5 (B) It must be $f(3)$ (C) It must be $f(5)$
 (D) It must be -5 (E) It cannot be determined

two one-sided limits match if $\lim_{x \rightarrow 3} f(x)$ exists, but we don't know if it does here.

5. If $-x^2 - x + 1 \leq g(x) \leq x^2 - x + 1$ for all $x \neq 0$, what is $\lim_{x \rightarrow 0} g(x)$?

[1]

- (A) 0 (B) 1 (C) 2
 (D) $g(0)$ (E) Cannot be determined

$\lim_{x \rightarrow 0} (-x^2 - x + 1) \leq \lim_{x \rightarrow 0} g(x) \leq \lim_{x \rightarrow 0} (x^2 - x + 1)$

$\rightarrow 1 \leq \lim_{x \rightarrow 0} g(x) \leq 1 \rightarrow \lim_{x \rightarrow 0} g(x) = 1$

6. Evaluate the following limit:

[1]

$$\lim_{x \rightarrow 4} \frac{x^2 - 8x + 16}{x - 4}$$

- (A) 0 (B) 8 (C) -8
(D) $+\infty$ (E) $-\infty$

$$\lim_{x \rightarrow 4} \frac{x^2 - 8x + 16}{x - 4} = \lim_{x \rightarrow 4} \frac{(x-4)^2}{x-4} = \lim_{x \rightarrow 4} (x-4) = 4-4 = \underline{\underline{0}}$$

7. If $\lim_{x \rightarrow 1} f(x) = 3$, $\lim_{x \rightarrow 1} g(x) = -2$, and $\lim_{x \rightarrow 1} h(x) = 4$, evaluate the limit

[1]

$$\lim_{x \rightarrow 1} \left(\frac{2f(x)}{g(x)} + \sqrt{h(x)} \right)$$

- (A) -1 (B) 3 (C) 13
(D) 5 (E) 7

$$\begin{aligned} \lim_{x \rightarrow 1} \left(\frac{2f(x)}{g(x)} + \sqrt{h(x)} \right) &= \frac{2 \lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x)} + \sqrt{\lim_{x \rightarrow 1} h(x)} = \frac{2 \cdot 3}{-2} + \sqrt{4} \\ &= -3 + 2 = \underline{\underline{-1}} \end{aligned}$$

(not 0)

8. If the function $f(x)$ is continuous on the interval $[-1, 3]$, $f(-1) = 1$, and $f(3) = 11$, which numbers below are guaranteed to be values of $f(x)$ by the Intermediate Value Theorem on the interval $(-1, 3)$? [1]

I. 3

II. $\sqrt{2}$ III. 3π

(A) I only (B) II only (C) III only

(D) I and II only (E) I, II, and III

all values between $f(-1)=1$ and $f(3)=11$
are guaranteed. $1 < \sqrt{2} < 3 < 3\pi < 11$,
so all 3 are guaranteed

9. Determine the value of the number k that makes the function $f(x)$ below continuous: [1]

$$f(x) = \begin{cases} 1 - kx & \text{if } x < 1, \\ k + x & \text{if } x \geq 1. \end{cases}$$

(A) 0 (B) 1 (C) $-3/4$ (D) $1/2$ (E) $15/17$

want $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) =$

$$\left. \begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (1 - kx) = 1 - k \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (k + x) = k + 1 \end{aligned} \right\} \rightarrow \begin{aligned} 1 - k &= k + 1 \\ 0 &= 2k \\ k &= \underline{\underline{0}} \end{aligned}$$

10. Consider the function

[1]

$$h(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x < 1, \\ x & \text{if } x > 1. \end{cases}$$

Which of the following are true?

- I. $\lim_{x \rightarrow 1^+} h(x)$ exists
 II. $\lim_{x \rightarrow 1^-} h(x)$ exists
 III. $\lim_{x \rightarrow 1} h(x)$ exists
 IV. $h(x)$ is continuous at $x = 1$

(A) I only

(B) I and II only

(C) I, II, and III only

(D) IV only

(E) I, II, III, and IV

$$\text{I. } \lim_{x \rightarrow 1^+} h(x) = \lim_{x \rightarrow 1^+} \frac{1}{x} = \frac{1}{1} = 1 \checkmark$$

$$\text{II. } \lim_{x \rightarrow 1^-} h(x) = \lim_{x \rightarrow 1^-} x = 1 \checkmark$$

$$\text{III. } \lim_{x \rightarrow 1^-} h(x) = \lim_{x \rightarrow 1^+} h(x) = 1, \text{ so } \lim_{x \rightarrow 1} h(x) = 1 \checkmark$$

IV. $h(1)$
not defined,
so h not
continuous
at $x=1$

11. Evaluate the following limit:

[1]

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2}}{x}$$

(A) $+\infty$ (B) $-\infty$

(C) 0

(D) 1

(E) -1

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2}}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2}}{x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2}}{1} \cdot \frac{1}{x}$$

$$= \lim_{x \rightarrow \infty} \sqrt{x^2 + 2} \cdot \frac{1}{\sqrt{x^2}}$$

$$= \lim_{x \rightarrow \infty} \sqrt{\frac{x^2 + 2}{x^2}} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{2}{x^2}}$$

12. The function $f(x) = \frac{x^2 + 1}{x^3 + 8}$ has which of the following? [1]

- (A) no vertical or horizontal asymptotes
- (B) 1 vertical asymptote and 1 horizontal asymptote**
- (C) 2 vertical asymptotes and 1 horizontal asymptote
- (D) 1 vertical asymptote and 2 horizontal asymptotes
- (E) 1 vertical asymptote and no horizontal asymptotes

• Vertical: $x^3 + 8 = 0 \rightarrow x^3 = -8 \rightarrow x = -2$ (and $x^2 + 1 = 5 \neq 0$ at $x = -2$, so Vert. asymptote there)

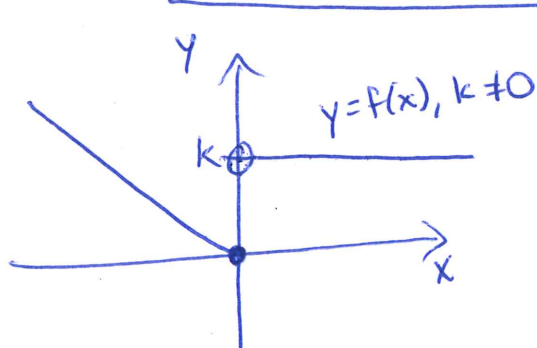
→ 1 vertical

• Horizontal: $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{x^2 + 1}{x^3 + 8} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$
 $= \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x} + \frac{1}{x^3}}{1 + \frac{8}{x^3}} = \frac{0 + 0}{1 + 0} = 0 \rightarrow$ horiz. asymp. $y = 0$
→ 1 horizontal

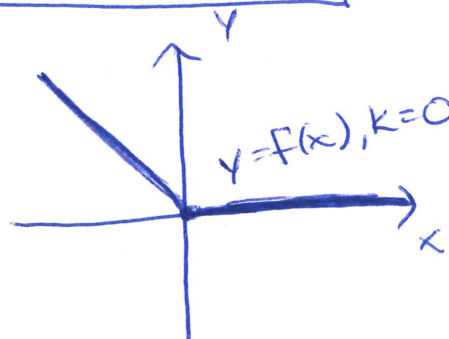
13. For what value of the number k is the following function differentiable at $x = 0$? [1]

$$f(x) = \begin{cases} -x & x \leq 0 \\ k & x > 0 \end{cases}$$

- (A) -2 (B) -1 (C) 0
- (D) 1 **(E) No value of k makes this function differentiable at $x = 0$**



not continuous at $x=0$,
so not differentiable



not locally linear
at $x=0$, so not differentiable

14. If $f(x) = 3x^{10}$, then $\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$ is which of the following? [1]
- (A) $f'(x)$ (B) $f'(1)$ (C) Does not exist
 (D) 0 (E) None of the above

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \underline{f'(1)} \text{ by limit def'n of derivative}$$

$$\left(f'(x) = 3 \cdot 10 \cdot x^9 = 30x^9, \text{ so } \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = f'(1) = 30 \cdot 1^9 = \underline{\underline{30}} \right)$$

15. If we want to calculate the derivative $f'(x)$ of $f(x) = 3x + 4$ using the limit definition of the derivative which of the following limits do we need to evaluate and to what does the limit evaluate? [1]

$$(A) \lim_{h \rightarrow 0} \frac{3(x+h) + 4 - (3x+4)}{h} = 3$$

$$(B) \lim_{h \rightarrow 0} \frac{3(x+h) + 4 - (3x+4)}{h} = 0$$

$$(C) \lim_{h \rightarrow 0} \frac{3h + 4 - (3x+4)}{h} = 3x + 3$$

$$(D) \lim_{h \rightarrow 0} \frac{3(x+h) + 4 - (3h+4)}{h} = 3$$

(E) None of the above.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3(x+h) + 4 - (3x+4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x + 3h + 4 - 3x - 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h}{h} = \lim_{h \rightarrow 0} 3 = \underline{\underline{3}} \end{aligned}$$

16. Below is the graph of the derivative $g'(x)$ of a function $g(x)$.

[1]

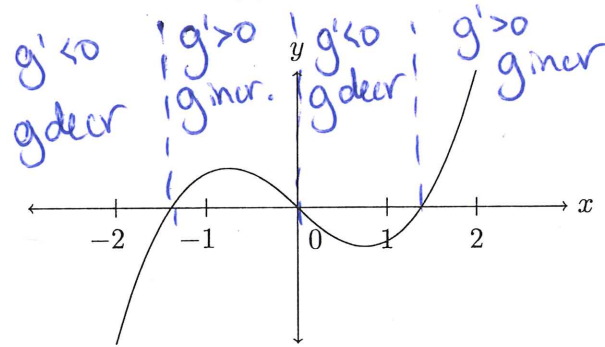
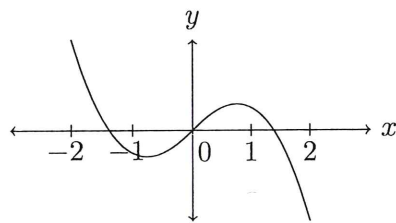


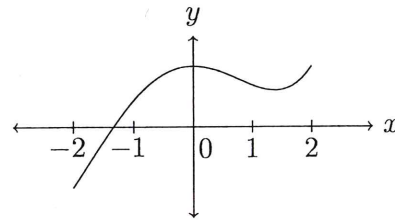
Figure 1: Graph of $g'(x)$.

Which of the following is a possible graph of $g(x)$?

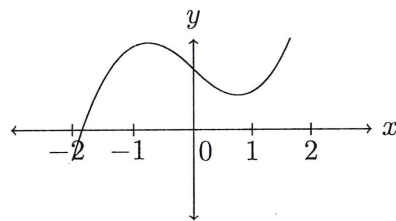
(A)



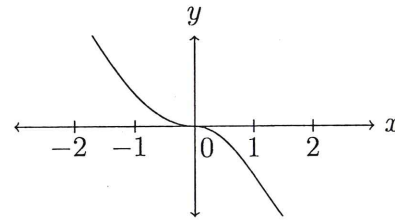
(B)



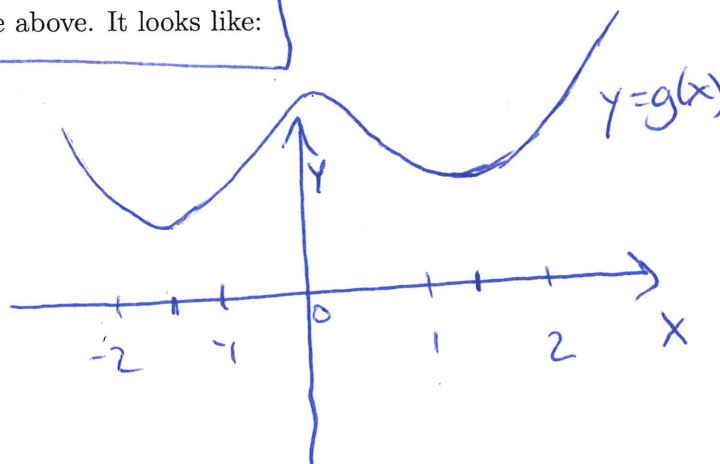
(C)



(D)



(E) None of the above. It looks like:



17. If $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ for $x > 0$, then $f'(4)$ is which of the following?

[1]

(A) $\frac{5}{4}$

(D) $\frac{3}{4}$

(C) $\frac{3}{16}$

(B) $\frac{255}{32}$

(E) $\frac{257}{32}$

$$f(x) = x^{1/2} + x^{-1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} + (-\frac{1}{2})x^{-3/2}$$

$$= \frac{1}{2} \left(\frac{1}{\sqrt{x}} - \frac{1}{x\sqrt{x}} \right)$$

$$= \frac{1}{2} \cdot \frac{x-1}{x\sqrt{x}}$$

$$f'(4) = \frac{1}{2} \cdot \frac{4-1}{4\sqrt{4}} = \underline{\underline{\frac{3}{16}}}$$

18. Determine $f'(1)$ for the function $f(x) = (x^3 - x^2 + 1)(x^4 - x + 2)$.

(A) 3 (B) 0 (C) 4

(D) 2 (E) 5

$$f'(x) = (3x^2 - 2x)(x^4 - x + 2) + (x^3 - x^2 + 1)(4x^3 - 1)$$

$$\rightarrow f'(1) = (3 - 2)(1 - 1 + 2) + (1 - 1 + 1)(4 - 1) = (1)(2) + (1)(3) = 5$$

19. Find the equation of the tangent line to the curve $y = \frac{x}{x+1}$ at $x = 1$.

(A) $y = \frac{1}{2}$ (B) $y = -\frac{1}{2}x + 1$ (C) $y = \frac{1}{2}x$

(D) $y = -\frac{1}{4}x + \frac{3}{4}$

(E) $y = \frac{1}{4}x + \frac{1}{4}$

$$y' = \frac{1(x+1) - x(1)}{(x+1)^2} \text{ at } x=1: y' = \frac{2-1}{2^2} = \frac{1}{4} \rightarrow \text{slope} = \frac{1}{4}$$

$$\text{point: } y(1) = \frac{1}{1+1} = \frac{1}{2}, \text{ so } (1, \frac{1}{2}) \rightarrow y - \frac{1}{2} = \frac{1}{4}(x - 1)$$

20. If $f(x) = \sin(x)$, determine $f^{(125)}(\pi)$.

(A) 1 (B) -1 (C) 0

(D) $1/2$ (E) $\sqrt{2}/2$

$$\rightarrow y = \frac{1}{4}x - \frac{1}{4} + \frac{1}{2}$$

$$\rightarrow y = \frac{1}{4}x + \frac{1}{4}$$

$$f^{(4)}(x) = f^{(2)}(x) = \dots = f^{(124)}(x) = \sin x, \text{ so}$$

$$f^{(125)}(x) = \frac{d}{dx} \sin x = \cos x. f^{(125)}(\pi) = \cos \pi = -1$$

21. To compute the derivative of $\sin^2 x$ with the chain rule by writing this function as a composition $f(g(x))$, what is the "inner" function $g(x)$?

(A) x (B) x^2 (C) $\sin x$

(D) $\sin^2 x$ (E) None of the above

$$\sin^2 x = (\sin x)^2, \text{ so } g(x) = \sin x, f(x) = x^2$$

22. Let $y = f(x)g(x)$. Using the table of values below, determine the value of $\frac{dy}{dx}$ when $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	5	2	4	4
2	3	4	1	3
3	2	3	2	2
4	4	1	5	5
5	1	5	3	1

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

$$\text{at } x=2: (4)(1) + (3)(3) = 4 + 9 = \boxed{13}$$

- (A) 9 (B) 12 (C) 13
 (D) 15 (E) 23

23. If $g(x) = \frac{ax+b}{cx+d}$, then $g'(1)$ is which of the following? Note: The numbers a, b, c , and d are constants.

(A) $\frac{a+b-c-d}{c+d}$ (B) $\frac{ad-bc}{(c+d)^2}$ (C) $\frac{a+b-c-d}{(c+d)^2}$

(D) $\frac{ad+bc}{c+d}$ (E) $\frac{ad+bc}{(c+d)^2}$

$$g'(x) = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2} \Rightarrow g'(1) = \frac{a(c+d) - (a+b)c}{(c+d)^2} = \frac{ac+ad-ac-bc}{(c+d)^2} = \frac{ad-bc}{(c+d)^2}$$

24. For the function $f(x) = x^3 \arctan(x)$, which of the following is $f'(1)$?

- (A) $\frac{3\pi}{4}$ (B) $\frac{3\pi}{4} + \frac{1}{2}$ (C) $\frac{1}{2}$
 (D) $\frac{\pi}{4}$ (E) $3 \tan(1) + \sec^2(1)$

$$f'(x) = 3x^2 \arctan(x) + x^3 \cdot \frac{1}{1+x^2}$$

$$\rightarrow f'(1) = 3(1)^2 \arctan(1) + (1)^3 \cdot \frac{1}{1+(1)^2} = 3\left(\frac{\pi}{4}\right) + \frac{1}{2}$$

25 Consider the functions $f(x) = \sin(x^2)$ and $g(x) = \sin^2(x)$. Which of the following is true?

- (A) $f'(x) = \cos(x^2)$ (B) $g'(x) = -2 \sin(x) \cos(x)$ (C) $f'(x) = g'(x)$
 (D) $f'(\pi) = g'(\pi) = 0$ (E) $f'(0) = g'(0)$

$$f'(x) = \cos(x^2) \cdot 2x$$

$$f'(\pi) = \cos(\pi^2) \cdot 2\pi \neq 0$$

$$f'(0) = \cos(0) \cdot 2(0) = 0$$

$$g'(x) = 2 \sin x \cos x$$

$$g'(\pi) = 2 \sin \pi \cos \pi = 0$$

$$g'(0) = 2 \sin 0 \cos 0 = 0$$

26 If $\frac{d}{dx}[f(4x)] = x^2$, then find $f'(x)$.

- (A) $\frac{x^2}{64}$ (B) $\frac{x^2}{16}$
 (D) x^2 (E) $4x^2$

$$\frac{d}{dx}[f(4x)] = f'(4x) \cdot 4 = x^2 \rightarrow f'(4x) = \frac{x^2}{4}$$

Let $u = 4x$.
 Then $\frac{u}{4} = x$, so $f'(u) = \frac{(\frac{u}{4})^2}{4} = \frac{\frac{u^2}{16}}{4} = \frac{u^2}{64}$

$$\rightarrow f'(x) = \frac{x^2}{64}$$

27. Give an equation of the tangent line to the curve $(x^2 + y^2)^2 = 4x^2y$ at the point $(1, 1)$.

- (A) $y = 1$ (B) $y = x$ (C) $y = 2x - 1$
 (D) $y = -x + 2$ (E) $y = -2x + 3$

$$\frac{d}{dx}[(x^2 + y^2)^2] = \frac{d}{dx}[4x^2y]$$

$$2(x^2 + y^2)(2x + 2y \frac{dy}{dx}) = 8xy + 4x^2 \frac{dy}{dx}$$

at $x=1, y=1$: $2(1+1)(2+2 \frac{dy}{dx}) = 8 + 4 \frac{dy}{dx}$

$$8 + 8 \frac{dy}{dx} = 8 + 4 \frac{dy}{dx}$$

$$4 \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = 0 \rightarrow \text{slope} = 0$$

point $(1, 1)$, slope = 0

$$\rightarrow y - 1 = 0(x - 1)$$

$$\rightarrow \underline{\underline{y = 1}}$$