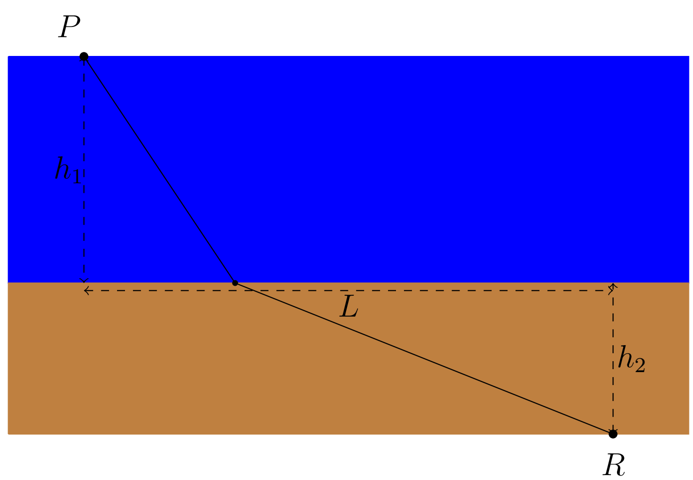
Math 1131 Applications: Optimization, II

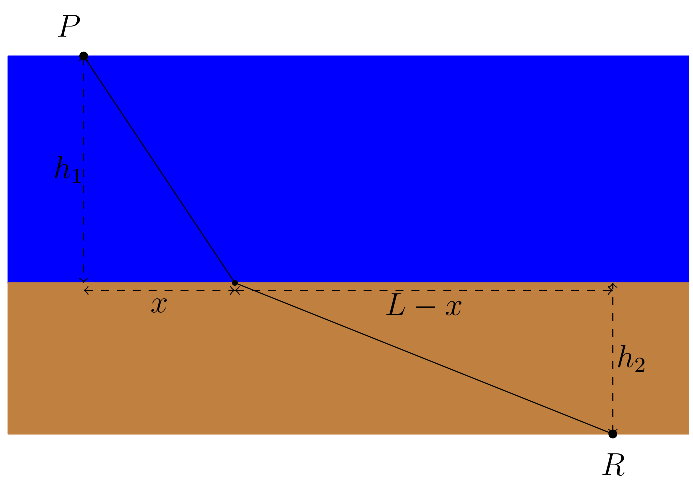
We discuss two more optimization problems, one about minimizing cost and the other about maximizing the length of a pole that fits around a corner.

Application 1. Cost of laying a pipeline.

A pipeline is to be laid from an oil platform P in water to a refinery R on land. See the bird's eye view in the figure below. The pipeline goes straight from P to a point on the shore and then straight from that point to R. Let P be h_1 miles to the shore, R be h_2 miles to the shore, L be the horizontal distance between P and R, $\$c_1$ per mile be the cost of laying pipeline under water and $\$c_2$ per mile be the cost of laying it on land. Where does the pipeline meet the shore to *minimize the total cost*?



To answer this, let the point on the shore nearest to P be at distance x miles from where the pipeline meets the shore, so $0 \leq x \leq L$. The figure below explains the meaning of x.



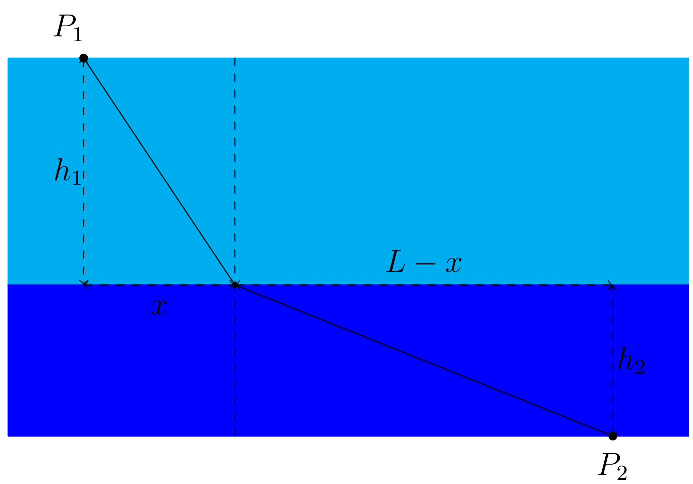
From the above figure, the length of pipeline in the water is $\sqrt{x^2+h_1^2}$ miles, so the cost of laying that much pipeline in water is $\$c_1\sqrt{x^2+h_1^2}$. Similarly, the cost of laying pipeline from the shore to the refinery is $\$c_2\sqrt{(L-x)^2+h_2^2}$. Therefore the total cost in dollars of laying the pipeline from P to R is

$$c_1\sqrt{x^2+h_1^2}+c_2\sqrt{(L-x)^2+h_2^2}. \tag{1}$$

We want to find the x in $[0, L]$ where this expression is minimal.

When we derived Snell's law in the first optimization applications handout, using the figure below, we wanted to minimize the total time for a light ray traveling at speed v_1 in air and v_2 in water to travel from P_1 in air to P_2 in water. We had worked out the total travel time of a light ray in air and water to be

$$\frac{1}{v_1}\sqrt{x^2+h_1^2}+\frac{1}{v_2}\sqrt{(L-x)^2+h_2^2}. \tag{2}$$



Comparing (1) and (2), they are mathematically similar even though the problems that led to them are different: minimizing the pipeline cost (1) for $0 \le x \le L$ is the same as minimizing the travel time (2) for $0 \le x \le L$ if we *replace* $1/v_1$ *with* c_1 *and* $1/v_2$ *with* c_2.

Remark. Why is "inverse velocity" of a light ray analogous to cost in the pipeline problem? This comes from the algebra linking velocity and cost to distance. If a light ray travels a distance d at speed v for time t then $d = vt$, so $t = d/v$. If a pipeline d miles long costs $\$c$ per mile to build then the pipeline's total cost is $\$cd$. When time for the light ray is analogous to cost for the pipeline, the time formula d/v is analogous to the cost formula cd, so $1/v$ is analogous to c.

In the first optimization applications handout, calculus showed us that there is exactly one x in $[0, L]$ where the travel time in (2) is minimized and it satisfies

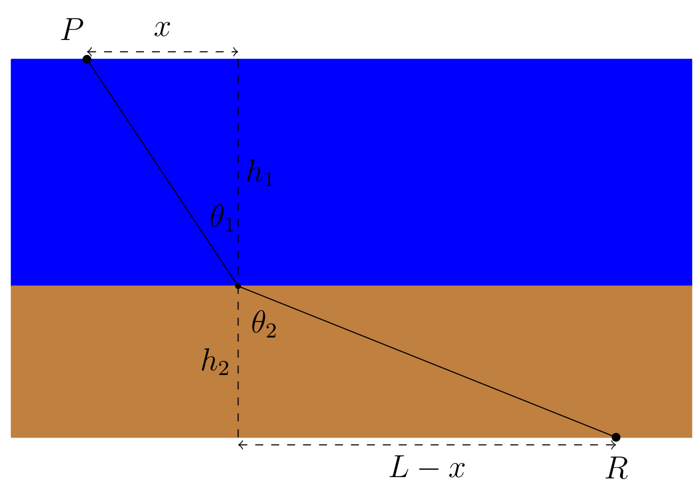
$$\frac{1}{v_1}\frac{x}{\sqrt{x^2+h_1^2}} = \frac{1}{v_2}\frac{L-x}{\sqrt{(L-x)^2+h_2^2}}.$$

By the same reasoning, one x in $[0, L]$ minimizes the total cost in (1) and it satisfies

$$c_1\frac{x}{\sqrt{x^2+h_1^2}} = c_2\frac{L-x}{\sqrt{(L-x)^2+h_2^2}}.$$

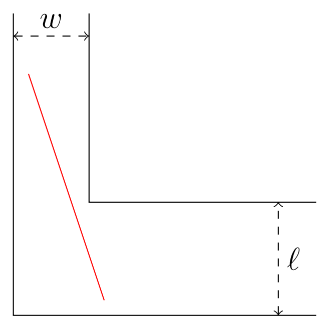
Interpreting $x/\sqrt{x^2+h_1^2}$ and $(L-x)/\sqrt{(L-x)^2+h_2^2}$ as sine functions of angles in the pipeline figure below, the cost-minimizing x fits Snell's law: if θ_1 is the pipeline angle in water and θ_2 is the pipeline angle on land, cost is minimized when the sine ratio is the inverse cost ratio:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{1/c_1}{1/c_2} = \frac{c_2}{c_1}.$$

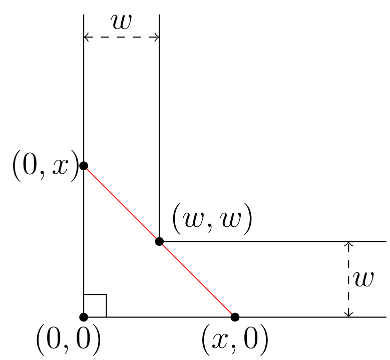


Application 2. Longest pole that can be moved around a corner.

In a hallway as in the figure below, what is the *longest* length for the red pole that allows it to be be moved around the corner? (To keep the problem 2-dimensional, the pole is moved at a fixed height above the ground.) The answer should depend on the two hallway widths w and ℓ.

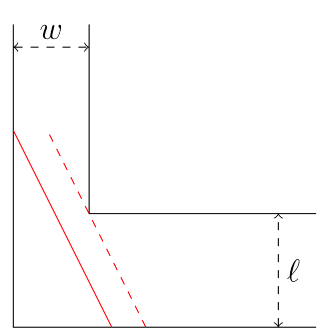


Example. Suppose the hallway widths are equal: $\ell = w$. Then it is intuitively plausible that the *longest* pole that fits around the hallway corner is the pole that forms the hypotenuse of an isosceles right triangle with legs on the outer walls when the pole touches the inner corner (the meeting point of the inner walls), as seen below.

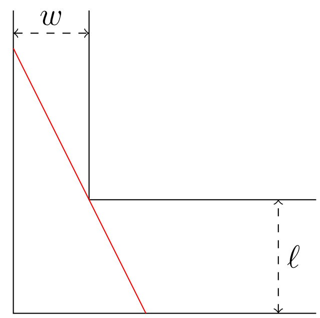


Using coordinates $(0, 0)$ for the outer corner, the inner corner is at (w, w), which is the center of the hypotenuse. Writing coordinates of the endpoints of the pole as $(x, 0)$ and $(0, x)$, the midpoint of the pole is $(x/2, x/2)$, so $x/2 = w$ and thus $x = 2w$. That makes the pole length $\sqrt{x^2 + x^2} = \sqrt{2}x = \boxed{2\sqrt{2}w}$.

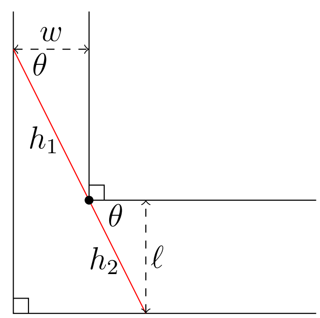
Now let's discuss the general case, when ℓ and w are arbitrary. If the pole is in a position that is not touching the inner corner (see solid red line below), we can move the pole parallel to itself until it touches that point (see dashed red line below), and



there is room to extend the length of the pole while still having it fit around the hallway corner. Therefore to make the pole as long as possible while being able to move it around the hallway corner, we only need to consider pole configurations that touch the inner corner and have ends on the outer walls (see below). The *shortest* such pole length will solve our problem. Therefore what sounded at first like a maximization problem is in fact a minimization problem!



In the figure below, let h_1 and h_2 be the lengths of the red line segment separated by the inner corner and let θ be the indicated angles (why are those angles equal?).



From the smaller right triangles we get $\cos\theta = w/h_1$ and $\sin\theta = \ell/h_2$, so the length of the red line segment is

$$h_1 + h_2 = \frac{w}{\cos\theta} + \frac{\ell}{\sin\theta} = w\sec\theta + \ell\csc\theta. \tag{3}$$

Call this length $L(\theta)$. We want to find the *minimum* value of $L(\theta)$ for $0 < \theta < \pi/2$. (At $\theta = 0$ or $\pi/2$, $L(\theta) = \infty$.) The first derivative $L'(\theta)$ is

$$\begin{aligned} (w\sec\theta + \ell\csc\theta)' &= w\sec\theta\tan\theta - \ell\csc\theta\cot\theta \\ &= w\frac{\sin\theta}{\cos^2\theta} - \ell\frac{\cos\theta}{\sin^2\theta} \\ &= \frac{w\sin^3\theta - \ell\cos^3\theta}{\cos^2\theta\sin^2\theta}. \end{aligned}$$

This vanishes at the angle θ where the numerator is 0: $w \sin^3\theta = \ell \cos^3\theta$. That is the same as $(\tan\theta)^3 = \ell/w$, or $\boxed{\theta = \arctan(\sqrt[3]{\ell/w})}$. As a reality check, when $\ell = w$ this angle θ is $\arctan(1) = \pi/4$, which matches the isosceles right triangle shape in the earlier example for the case $\ell = w$.

To check the angle in $(0, \pi/2)$ where $L'(\theta) = 0$ is a minimum for $L(\theta)$, we check $L''(\theta) > 0$ at that angle. For all angles θ,

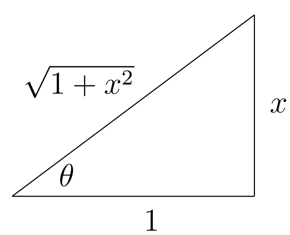
$$\begin{aligned} L''(\theta) &= (L'(\theta))' \\ &= w(\sec\theta\tan\theta)' - \ell(\csc\theta\cot\theta)' \\ &= w((\sec\theta\tan\theta)\tan\theta + (\sec\theta)\sec^2\theta) - \ell((-\csc\theta\cot\theta)\cot\theta + (\csc\theta)(-\csc^2\theta)) \\ &= w(\sec\theta\tan^2\theta + \sec^3\theta) + \ell(\csc\theta\cot^2\theta + \csc^3\theta). \end{aligned}$$

When $0 < \theta < \pi/2$, all the standard trigonometric functions are positive, and also w and ℓ are positive, so $L''(\theta) > 0$ for all θ in $(0, \pi/2)$.

It remains to compute $L(\theta)$ when $\theta = \arctan(\sqrt[3]{\ell/w})$:

$$\begin{aligned} L(\theta) &= w\sec(\arctan(\sqrt[3]{\ell/w})) + \ell\csc(\arctan(\sqrt[3]{\ell/w})) \\ &= \frac{w}{\cos(\arctan(\sqrt[3]{\ell/w}))} + \frac{\ell}{\sin(\arctan(\sqrt[3]{\ell/w}))} \\ &= ?? \end{aligned}$$

What are $\cos(\arctan x)$ and $\sin(\arctan x)$ when $x > 0$?



From the triangle above, where $\theta = \arctan x$, we have

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}, \quad \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}.$$

Therefore

$$\begin{aligned}\frac{w}{\cos(\arctan x)} + \frac{\ell}{\sin(\arctan x)} &= \frac{w}{1/\sqrt{1+x^2}} + \frac{\ell}{x/\sqrt{1+x^2}} \\ &= w\sqrt{1+x^2} + \frac{\ell\sqrt{1+x^2}}{x}.\end{aligned}$$

Setting $x = \sqrt[3]{\ell/w}$, we get

$$\begin{aligned}w\sqrt{1+x^2} + \frac{\ell\sqrt{1+x^2}}{x} &= w\sqrt{1+(\ell/w)^{2/3}} + \frac{\ell\sqrt{1+(\ell/w)^{2/3}}}{\sqrt[3]{\ell/w}} \\ &= \frac{w\sqrt{w^{2/3}+\ell^{2/3}}}{\sqrt[3]{w}} + \frac{\ell\sqrt{w^{2/3}+\ell^{2/3}}}{\sqrt[3]{w}\,\sqrt[3]{\ell/w}} \\ &= w^{2/3}\sqrt{w^{2/3}+\ell^{2/3}} + \ell^{2/3}\sqrt{w^{2/3}+\ell^{2/3}} \\ &= (w^{2/3}+\ell^{2/3})\sqrt{w^{2/3}+\ell^{2/3}} \\ &= \boxed{(w^{2/3}+\ell^{2/3})^{3/2}}.\end{aligned}$$

As a reality check, when $\ell = w$ we have $(w^{2/3}+\ell^{2/3})^{3/2} = (2w^{2/3})^{3/2} = 2^{3/2}w = 2\sqrt{2}w$, and that is the pole length we found earlier in the example when $\ell = w$.