§15.9 Change of Variables Practice Exercises

1. Find the Jacobian for each transformation.
   (a) $x = 5u - v$, $y = u + 3v$
   (b) $x = uv$, $y = u/v$
   (c) $x = e^{-r}\sin\theta$, $y = e^{r}\cos\theta$

2. Find the image of the set S under the given transformation.
   (a) S is the square bounded by the lines $u = 0$, $u = 3$, $v = 0$, $v = 3$; $x = 2u + 3v$, $y = u - v$
   (b) S is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$; $x = u^2$, $y = v$

3. Use the transformation given by $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$ to compute the double integral $\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$.

Solutions

1. Find the Jacobian for each transformation.

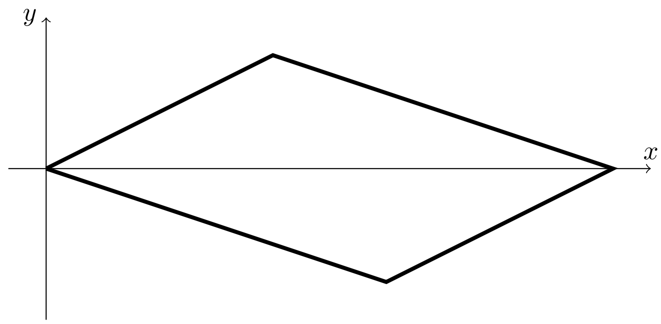
   (a) $\begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = 15 - (-1) = 16$

   (b) $\begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = (-u/v) - u/v = -2u/v$

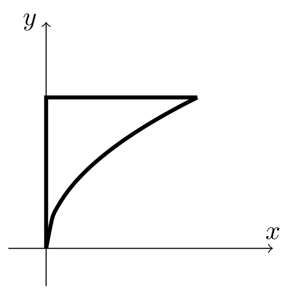
   (c) $\begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix} = \sin^2\theta - \cos^2\theta$

2. Find the image of the set S under the given transformation.

   (a) Vertices $(0,0)$, $(3,0)$, $(3,3)$ and $(0,3)$ in xy-plane are mapped to $(0,0)$, $(6,3)$, $(15,0)$ and $(9,-3)$, respectively, in the uv-plane:



   (b) S is bounded by $u = 0$, $v = 1$, and $u = v$
   $u = 0 \Rightarrow x = 0$ and $y = v$ (we get $x = 0$ as a bounding curve)
   $v = 1 \Rightarrow x = u^2$ and $y = 1$ (we get $y = 1$ as a bounding curve)
   $u = v \Rightarrow x = v^2$ and $y = v \Rightarrow y = \sqrt{x}$



3. $x = \frac{1}{4}(u+v) \Rightarrow 4x = u + v \Rightarrow v = 4x - u.$

$y = \frac{1}{4}(v - 3u) \Rightarrow 4y = v - 3u \Rightarrow 4y = 4x - u - 3u \Rightarrow u = x - y$

Then $v = 4x - (x - y) = 3x + y.$

The vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$ in the xy-plane are mapped to $(-4,0)$, $(4,0)$, $(4,8)$, and $(-4,8)$, respectively, in the uv-plane (giving the rectangle $-4 \leqslant u \leqslant 4$, $0 \leqslant v \leqslant 8$).

The Jacobian is $\begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = 1/16 - (-3/16) = 1/4$

$$\begin{aligned}
\iint_R (4x + 8y)\, dA &= \int_{-4}^{4} \int_{0}^{8} (u + v + 2v - 6u) \tfrac{1}{4}\, dv\, du \\
&= \tfrac{1}{4} \int_{-4}^{4} \left[\tfrac{3v^2}{2} - 5uv \right]_0^8 du \\
&= \tfrac{1}{4} \int_{-4}^{4} 96 - 40u\, du \\
&= \tfrac{1}{4} \left[96u - 20u^2 \right]_{-4}^{4} du \\
&= \tfrac{1}{4} \left[(384 - 320) - (-384 - 320) \right] = 192
\end{aligned}$$