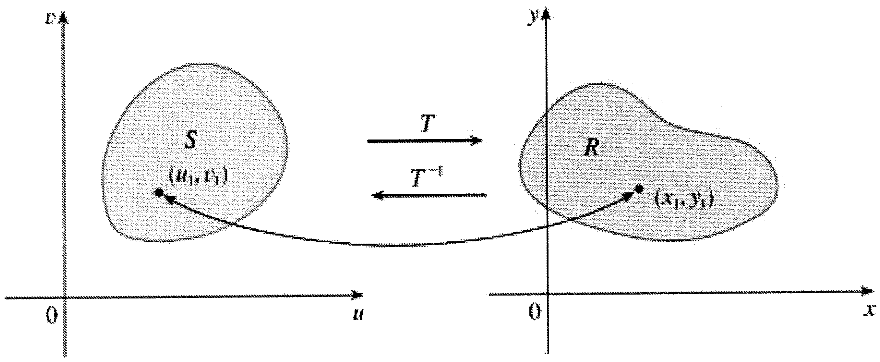
§15.9 Change of Variables in Multiple Integrals

In Calculus I, a useful technique to evaluate many difficult integrals is by using a u-substitution, which is essentially a change of variable to simplify the integral. Sometimes changing variables can make a huge difference in evaluating a double integral as well, as we have seen already with polar coordinates. This is often a helpful technique for triple integrals as well.

In general, say that we have a transformation $T(u, v) = (x, y)$ that maps a region S to a region R (see picture below). All images are taken from Stewart, 8th Edition.



We define the **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ as

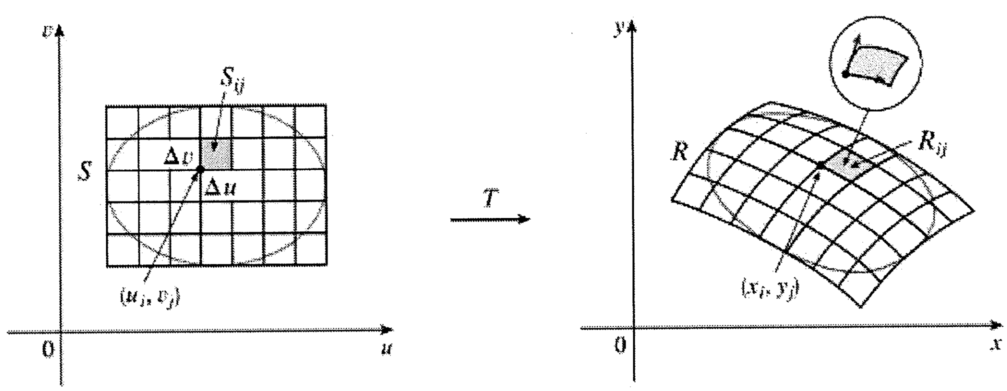
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

We can use this notation to approximate the subareas ΔA of the region R, the image of S under T:

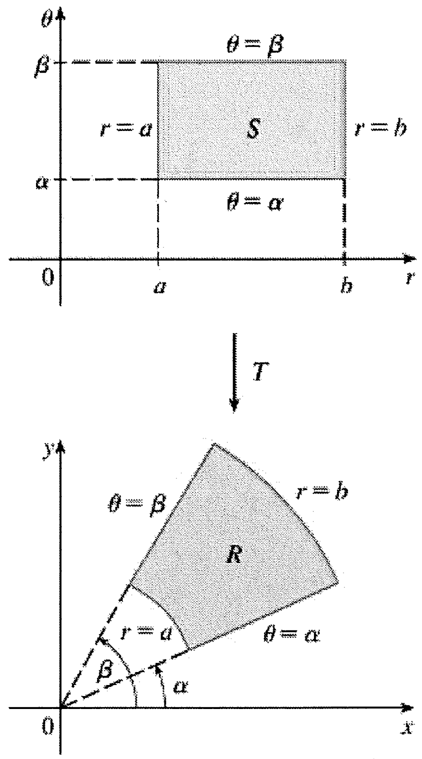
$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v.$$

Dividing the region S in the uv-plane into rectangles S_{ij} and calling their images in the xy-plane R_{ij} (see picture below), we can approximate the double integral of a function $f(x, y)$. Taking limits of the double sum, we get the following:

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(g(u, v), h(u, v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, du\, dv$$



We have seen an example of this with polar coordinates. In that case, the transformation $T(r,\theta) = (x,y)$ is given by $x = g(r,\theta) = r\cos\theta$ and $y = h(r,\theta) = r\sin\theta$.



The Jacobian of the transformation T is given by

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, we have that

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_S f(r\cos\theta, r\sin\theta)\left|\tfrac{\partial(x,y)}{\partial(r,\theta)}\right|\,dr\,d\theta = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

One way to understand the extra factor of r in the integral is to think about how the area of each region is affected if we change the bounds. If we keep the bounds on θ the same, say $\alpha \leqslant \theta \leqslant \beta$, but change the radius from $1 \leqslant r \leqslant 2$ to $101 \leqslant r \leqslant 102$, the area of the region in terms of x and y dramatically increases, even though the area of the rectangle in r and θ would be the same. In short, the bigger the radius, the bigger the area, so the area is scaled up accordingly.

The Jacobian is defined in a similar manner for a transformation with three variables, say $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$. Then we have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In particular, there are similar factors with cylindrical coordinates and spherical coordinates, two examples of three-variable transformations, which have Jacobians of r and $\rho^2\sin\phi$, respectively.

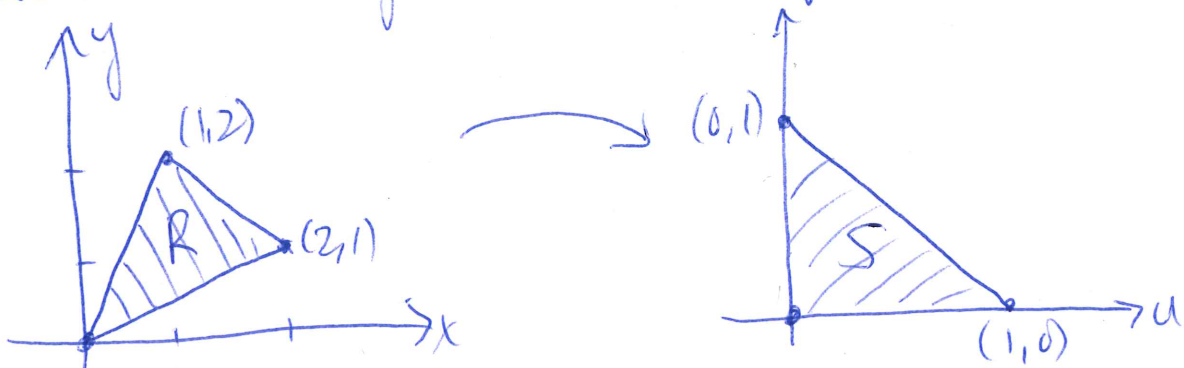
One of the most useful applications of a change of variables is simplifying otherwise complicated and/or tedious integrals. One way to do this is to look at the boundary curves of the region R and see where they are taken under the transformation T. Looking at the boundary of R allows us to determine the region S and use the Jacobian to compute the integral in a different way.

Example 1: Use the transformation given by $x = 2u + v$, $y = u + 2v$ to compute the double integral $\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$.

First, note that $v = x - 2u = x - 2(y - 2v) = x - 2y + 4v \Rightarrow v = -\frac{1}{3}(x - 2y)$

$\therefore u = y - 2v = y + \frac{2}{3}(x - 2y) = \frac{1}{3}(2x - y)$.

What is the region S?



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \text{ so}$$

$$\iint_R (x - 3y)\, dA = \iint_S [(2u+v) - 3(u+2v)] \cdot 3\, du\, dv$$

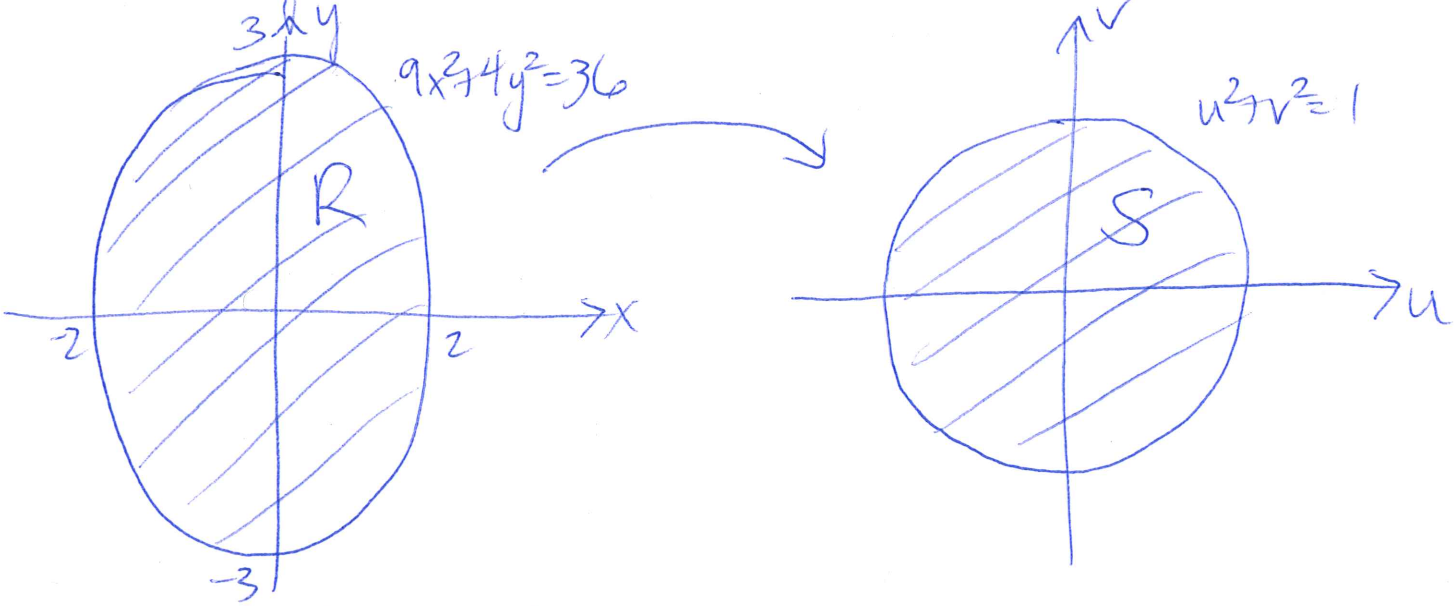
$$= \int_0^1 \int_0^{1-u} 3(-u - 5v)\, dv\, du = -3 \int_0^1 \left(uv + \frac{5}{2}v^2\right)\Big|_0^{1-u} du$$

$$= -3 \int_0^1 \left[\overbrace{u(1-u)}^{u-u^2} + \frac{5}{2}(1-u)^2\right] du = -3\left(\frac{1}{2}u^2 - \frac{1}{3}u^3 - \frac{5}{6}(1-u)^3\right)\Big|_0^1$$

$$= -3\left[\left(\frac{1}{2} - \frac{1}{3} - 0\right) - \left(0 - 0 - \frac{5}{6}\right)\right] = \underline{-3}.$$

Example 2: Use the transformation given by $x = 2u$, $y = 3v$ to compute the double integral $\iint\limits_R x^2 \, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$.

$$36 = 9x^2 + 4y^2 = 9(4u^2) + 4(9v^2) = 36u^2 + 36v^2 \Rightarrow u^2 + v^2 = 1$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\therefore \iint_R x^2 \, dA = \iint_S (2u)^2 \cdot 6 \, dA$$

$$= \int_0^{2\pi} \int_0^1 24 r^2 \cos^2\theta \cdot r \, dr \, d\theta$$

$$= 24 \left(\int_0^{2\pi} \cos^2\theta \, d\theta \right) \left(\int_0^1 r^3 \, dr \right) = 24 \cdot \pi \cdot \frac{1}{4} = \underline{6\pi}$$