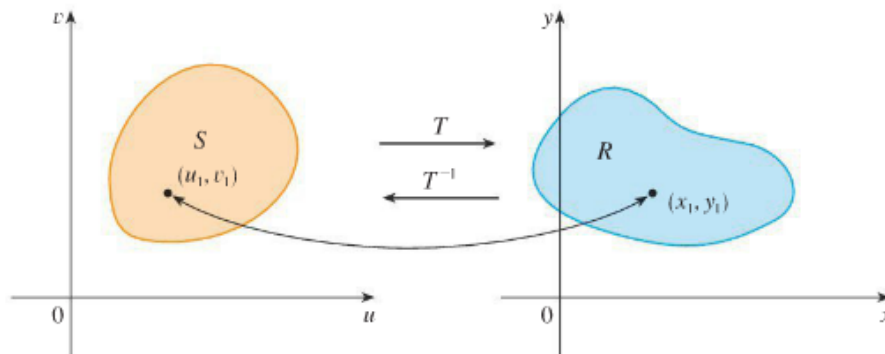


§15.9 Change of Variables in Multiple Integrals

In Calculus I, a useful technique to evaluate many difficult integrals is by using a u -substitution, which is essentially a change of variable to simplify the integral. Sometimes changing variables can make a huge difference in evaluating a double integral as well, as we have seen already with polar coordinates. This is often a helpful technique for triple integrals as well.

In general, say that we have a transformation $T(u, v) = (x, y)$ that maps a region S to a region R (see picture below). All images are taken from Stewart, 8th Edition.



We define the **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ as

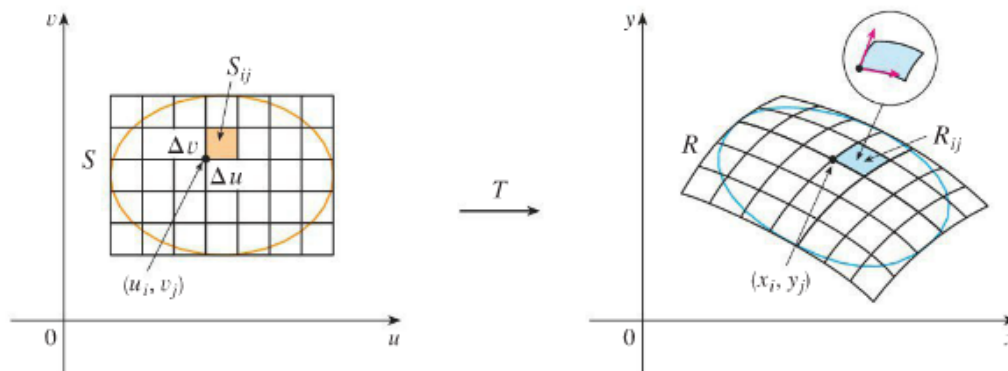
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$$

We can use this notation to approximate the subareas ΔA of the region R , the image of S under T :

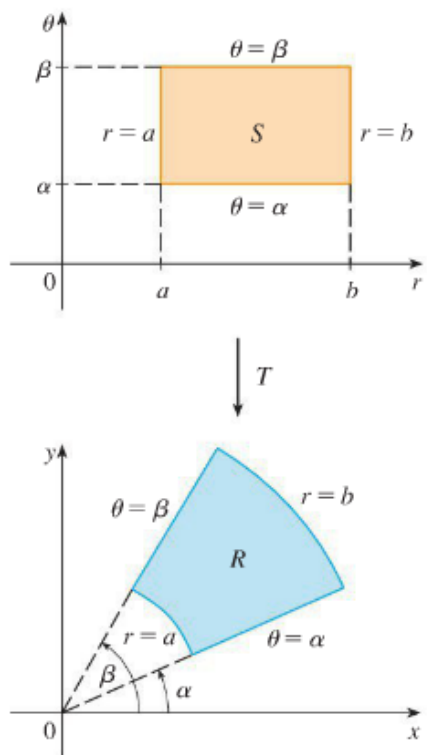
$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v.$$

Dividing the region S in the uv -plane into rectangles S_{ij} and calling their images in the xy -plane R_{ij} (see picture below), we can approximate the double integral of a function $f(x, y)$. Taking limits of the double sum, we get the following:

$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$



We have seen an example of this with polar coordinates. In that case, the transformation $T(r, \theta) = (x, y)$ is given by $x = g(r, \theta) = r \cos \theta$ and $y = h(r, \theta) = r \sin \theta$.



The Jacobian of the transformation T is given by

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

Therefore, we have that

$$\iint_R f(x, y) \, dx \, dy = \iint_S f(r \cos \theta, r \sin \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr \, d\theta = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

One way to understand the extra factor of r in the integral is to think about how the area of each region is affected if we change the bounds. If we keep the bounds on θ the same, say $\alpha \leq \theta \leq \beta$, but change the radius from $1 \leq r \leq 2$ to $101 \leq r \leq 102$, the area of the region in terms of x and y dramatically increases, even though the area of the rectangle in r and θ would be the same. In short, the bigger the radius, the bigger the area, so the area is scaled up accordingly.

The Jacobian is defined in a similar manner for a transformation with three variables, say $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$. Then we have

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In particular, there are similar factors with cylindrical coordinates and spherical coordinates, two examples of three-variable transformations, which have Jacobians of r and $\rho^2 \sin \phi$, respectively.

One of the most useful applications of a change of variables is simplifying otherwise complicated and/or tedious integrals. One way to do this is to look at the boundary curves of the region R and see where they are taken under the transformation T . Looking at the boundary of R allows us to determine the region S and use the Jacobian to compute the integral in a different way.

Example 1: Use the transformation given by $x = 2u + v$, $y = u + 2v$ to compute the double integral

$$\iint_R (x - 3y) \, dA, \text{ where } R \text{ is the triangular region with vertices } (0, 0), (2, 1), \text{ and } (1, 2).$$

Example 2: Use the transformation given by $x = 2u$, $y = 3v$ to compute the double integral $\iint_R x^2 dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$.