

§14.8 Lagrange Multipliers Practice Exercises

1. Find the absolute maximum and minimum values of the function $f(x, y) = y^2 - x^2$ over the region given by $x^2 + 4y^2 \leq 4$. (Hint: use Lagrange multipliers to find the max and min on the boundary)
2. Find the maximum area of a rectangle with sides measuring x and y if the perimeter is 14. Is there a minimum value of the area?

Solutions

1. $f_x = -2x = 0 \Rightarrow x = 0$

$$f_y = 2y = 0 \Rightarrow y = 0$$

$(0, 0)$ is a critical point inside the given region

Constraint is $g(x, y) = x^2 + 4y^2 = 4$. Lagrange multipliers: $\vec{\nabla} f = \lambda \vec{\nabla} g$.

$$\begin{aligned} -2x &= \lambda 2x \\ 2y &= \lambda 8y \\ x^2 + 4y^2 &= 4 \end{aligned}$$

If $x \neq 0$ and $y \neq 0$, then $\lambda = -1$ and $\lambda = \frac{1}{4}$, impossible, so must have $x = 0$ or $y = 0$.

If $x = 0$ and $x^2 + 4y^2 = 4$, then $y = \pm 1$, so $(0, 1)$ and $(0, -1)$ are possible points.

If $y = 0$ and $x^2 + 4y^2 = 4$, then $x = \pm 2$, so $(2, 0)$ and $(-2, 0)$ are possible points.

$$f(0, 0) = 0$$

$$f(0, 1) = 1 \text{ (absolute maximum value)}$$

$$f(0, -1) = 1 \text{ (absolute maximum value)}$$

$$f(2, 0) = -4 \text{ (absolute minimum value)}$$

$$f(-2, 0) = -4 \text{ (absolute minimum value)}$$

2. Area = $A(x, y) = xy$

The only critical point is $(0, 0)$, but it doesn't make sense in the context of the problem, so it can be ignored.

Constraint is perimeter $P(x, y) = 2x + 2y = 14$. Lagrange multipliers: $\vec{\nabla} A = \lambda \vec{\nabla} P$.

$$\begin{aligned} y &= 2\lambda \\ x &= 2\lambda \\ 2x + 2y &= 14 \end{aligned}$$

We have $x = 2\lambda = y$ so $x + x = 7$ and $x = \frac{7}{2}$, so $y = \frac{7}{2}$.

$$A\left(\frac{7}{2}, \frac{7}{2}\right) = \frac{49}{4}$$

Pick any other point (x, y) satisfying the constraint, for example $(1, 6)$: $A(1, 6) = 6 < \frac{49}{4}$. Thus, $\frac{49}{4}$ is the maximum value. There is no minimum value if $x \neq 0$ and $y \neq 0$.