§14.8 Lagrange Multipliers

In Calculus I, we first learned how to find and classify critical points, which allow us to find the location of local maxima and minima. We also discussed the application of finding the absolute maximum and minimum values of a function $y = f(x)$ over a closed interval $[a, b]$. Recall that the absolute maximum and minimum can only occur at either critical points or at the endpoints of the interval but nowhere else, assuming that $f(x)$ is continuous on $[a, b]$. This is known as the **Extreme Value Theorem**, and our goal is to now extend these ideas to functions of two or more variables.

Once again, it should be clear that an absolute maximum or minimum value can occur at a critical point, but what corresponds to the endpoints? First, absolute extrema are only guaranteed to exist over a set D that is both *closed* and *bounded*. What does that mean? A set D is called closed if it includes its boundary. For example, the set defined by $x^2 + y^2 \leqslant 1$ is closed, but $x^2 + y^2 < 1$ is not (the latter set would have a dashed circle for its boundary while the first would be solid). A set D is called bounded if it doesn't extend infinitely in any direction. Mathematically, this is equivalent to being able to fit the set D inside of a disk with some finite radius.

We can now state the Extreme Value Theorem for a function of two variables. If $f(x, y)$ is continuous on a closed, bounded set D in the plane, then f will attain an absolute maximum and minimum value at either a critical point inside D or on the boundary of the set D.

In the event that the boundary curve of a set D is given by one equation, the computations can still be tedious, but we can use another method to find the absolute maximum and minimum values. This curve can be thought of as a level curve of some function $g(x, y)$, so we write the curve as $g(x, y) = k$ (for some constant k). This is called a **constraint**, and solving many real-world problems goes hand-in-hand with solving constrained optimization problems.

The method that we will use is called the method of **Lagrange multipliers**, which essentially relies on the observation that a function $f(x, y)$ is maximized or minimized over a curve $g(x, y) = k$ whenever the functions' gradient vectors are parallel at a point on the curve (see the images in Section 14.8 of our textbook for a nice illustration of this). This works for functions defined with any number of variables, but we will state the system in the case of two variables. Namely, given a function $f(x, y)$ that we wish to maximize or minimize and a constraint $g(x, y) = k$, we seek solutions to the system of equations

$$\begin{aligned} \vec{\nabla} f(x, y) &= \lambda \cdot \vec{\nabla} g(x, y) \\ g(x, y) &= k \end{aligned}$$

Here, λ is a constant, called a Lagrange multiplier. Also, note that $\vec{\nabla} f = \lambda \vec{\nabla} g$ is really *two* equations: $f_x = \lambda g_x$ and $f_y = \lambda g_y$, which come from the corresponding components of the gradient vectors.

So, for a function $f(x, y)$ with constraint curve $g(x, y) = k$, we really wish to solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = k \end{cases}$$

Example 1: Find the absolute maximum and minimum values of $f(x, y) = xy^2$ on $x^2 + y^2 = 3$.

We want to solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases} \Rightarrow \begin{cases} y^2 = \lambda \cdot 2x \\ 2xy = \lambda \cdot 2y \\ x^2 + y^2 = 3 \end{cases}$$

Note: λ is the same constant in both equations!

If $y^2 = \lambda \cdot 2x$, then $x = y = 0$ is a solution, but $(0,0)$ is not on the curve $x^2+y^2=3$. So, we can write $\lambda = \frac{y^2}{2x}$.

If $2xy = \lambda \cdot 2y$, then either $y=0$ or $\lambda = x$.

$y = 0 \Rightarrow x^2 + 0 = 3 \Rightarrow x = \pm\sqrt{3} \Rightarrow \underline{(\sqrt{3}, 0), (-\sqrt{3}, 0)}$

$\lambda = x \Rightarrow \underbrace{\lambda = x = \frac{y^2}{2x}}_{\text{same } \lambda \text{ for both}} \Rightarrow y^2 = 2x^2$

Substituting $y^2 = 2x^2$ into $x^2+y^2=3$, we get

$x^2 + 2x^2 = 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow$ <u>four points $(\pm 1, \pm\sqrt{2})$</u>

Plugging these into our function $f(x,y) = xy^2$:

$f(\pm\sqrt{3}, 0) = 0, \quad f(1, \pm\sqrt{2}) = 2, \quad f(-1, \pm\sqrt{2}) = -2$

$\therefore$ <u>Absolute max value is 2, absolute min value is -2</u>

has no critical points

Example 2: Find the maximum and minimum values of $\underbrace{f(x,y) = 3x+y}$ subject to the constraint $x^2+y^2 = 10$.

$$\begin{cases} 3 = \lambda \cdot 2x \Rightarrow \lambda = \frac{3}{2x} \\ 1 = \lambda \cdot 2y \Rightarrow \lambda = \frac{1}{2y} \\ x^2+y^2=10 \end{cases}$$

$$\Rightarrow \frac{3}{2x} = \frac{1}{2y} \Rightarrow 3y = x$$

Combining $3y=x$ and $x^2+y^2=10$, we have

$$x^2+y^2 = (3y)^2 + y^2 = 10y^2 = 10 \Rightarrow y = \pm 1$$

Since $3y=x$, we consider the points $(3,1)$ & $(-3,-1)$.

Evaluating $f(x,y) = 3x+y$ at these points gives

$$f(3,1) = 10 \quad \text{and} \quad f(-3,-1) = -10.$$

$\therefore$ Absolute max value is 10, absolute min value is -10

Note: this curve is not bounded, so it may not have an absolute max/min!

Sometimes we only obtain one value from this method, meaning that there is either a maximum with no minimum or vice versa. It is then important to check analytically to determine which one has been found.

Example 3: Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$.

$$\begin{cases} 2x = \lambda \cdot y \\ 2y = \lambda \cdot x \\ xy = 1 \end{cases}$$

Either $x = y = 0 \Rightarrow (0,0)$ (not on $xy=1$) or $\lambda = \frac{2x}{y} = \frac{2y}{x}$

$\Rightarrow x^2 = y^2$

$x^2 = y^2 \Rightarrow y = \pm x$, but since we need the product of x and y to be positive (points must be on $xy=1$), we can ignore the line $y = -x$.

If $xy = 1$ and $y = x$, then we have $x^2 = 1$, which yields the points $(1,1)$ and $(-1,-1)$.

$f(1,1) = f(-1,-1) = 2$, but is this an absolute max or min value?

Pick any other point on $xy=1$ to check.

For example, $(2, \frac{1}{2})$ gives $f(2, \frac{1}{2}) = 4 + \frac{1}{4} > 2$, so the absolute min value is 2.

There is no absolute max value, which you can see by observing points of the form $(n, \frac{1}{n})$.