§14.5 The Chain Rule

The One-Variable Chain Rule

First, a quick reminder about the Chain Rule that you saw in Calculus I. Say that we have a function $y = f(g(x))$. Then, using two different notations, we can find the derivative of y as

$$y' = f'(g(x)) \cdot g'(x) \qquad \text{or} \qquad \frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

Example 1: If $y = (x^2+1)^3$, determine y'.

$$y' = 3(x^2+1)^2 \cdot 2x = \underline{6x(x^2+1)^2}$$

Example 2: If $f(x) = e^{\cos(x^4+x)}$, determine $f'(x)$.

$$\underline{f'(x) = e^{\cos(x^4+x)} \cdot -\sin(x^4+x) \cdot (4x^3+1)}$$

What if there are multiple dependent variables?

For example, say that $z = f(x,y)$, but we also have that $x = x(t)$ and $y = y(t)$ (that is, x and y are both functions of t). Ultimately, this means that $z = z(t)$, where x and y are "intermediate" variables of a sort, so it should make sense to find the derivative of z with respect to t. But how do we compute it? First, it is helpful to sketch and keep in mind a quick tree diagram like the one below:

$$\begin{tikzcd}
 & z \arrow[dl] \arrow[dr] & \\
x \arrow[d] & & y \arrow[d] \\
t & & t
\end{tikzcd}$$

In order to find dz/dt, we need to add up all of the possible derivatives with respect to t, namely we want to follow every branch that ends in t and add those derivatives. Therefore, we have that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Note that two of the derivatives present are partial derivatives, since z is a function of more than one variable, but the others are not since x and y are only functions of one variable, t.

Example 3: Determine dz/dt if $z = x \ln y$, $x = \cos t$, and $y = e^{2t}$.

$$\frac{dz}{dt} = (\ln y)(-\sin t) + \left(\frac{x}{y}\right)(2e^{2t})$$

Example 4: Determine the value of dz/dt at $t = 1$ if $z = \dfrac{xy^2}{x+1}$, $x = t^2 - 1 + \ln t$, and $y = t\cos(\pi t)$.

If $t=1$, then $x=0$ and $y=-1$.

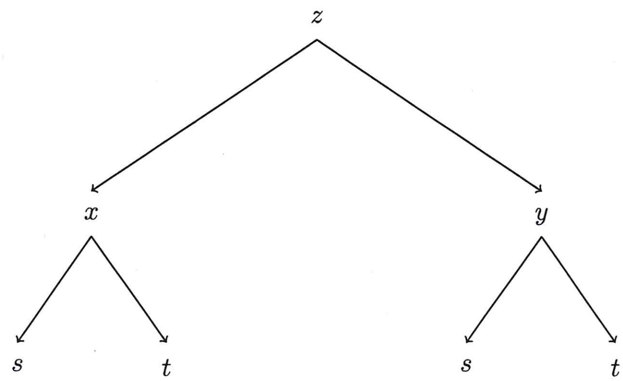
$$\frac{dz}{dt} = y^2\left(\frac{(x+1)-x}{(x+1)^2}\right)\cdot(2t+\ln t) + \left(\frac{2xy}{x+1}\right)(\cos(\pi t) - t\sin(\pi t)\cdot\pi)$$

$$\therefore \left.\frac{dz}{dt}\right|_{t=1} = (-1)^2\frac{1}{(0+1)^2}(2+0) + (0)(-1-0) = \underline{3}$$

What if there are multiple dependent and independent variables?

Say, for example, that $z = f(x, y)$ but $x = x(s, t)$ and $y = y(s, t)$. z is a function of more than one variable, but so are both x and y. z is ultimately a function of both s and t, so it now makes sense to take the derivative of z with respect to either s or t. How do we compute the partial derivatives?

Just like before, sketch a tree diagram and follow all paths that lead to the desired variable and add up all possible derivatives that correspond to each path.



Therefore, in this circumstance, we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \text{and} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 5: If $z = x^2y^3$, $x = s\cos t$, and $y = s\sin t$, determine $\partial z/\partial s$ and $\partial z/\partial t$.

$$\frac{\partial z}{\partial s} = (2xy^3)(\cos t) + (3x^2y^2)(\sin t)$$

$$\frac{\partial z}{\partial t} = (2xy^3)(-s \cdot \sin t) + (3x^2y^2)(s \cdot \cos t)$$

We can also have more variables than in any of these examples. However, the method remains the same. Draw a quick tree diagram and make sure to add up all possible derivatives along any branches that end in the desired variable.

Example 6: Find all possible first partials of $z = x^4 + x^2y$ if $x = s + 2t - u$ and $y = stu^2$.

$$\frac{\partial z}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(tu^2)$$

$$\frac{\partial z}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(su^2)$$

$$\frac{\partial z}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2stu)$$