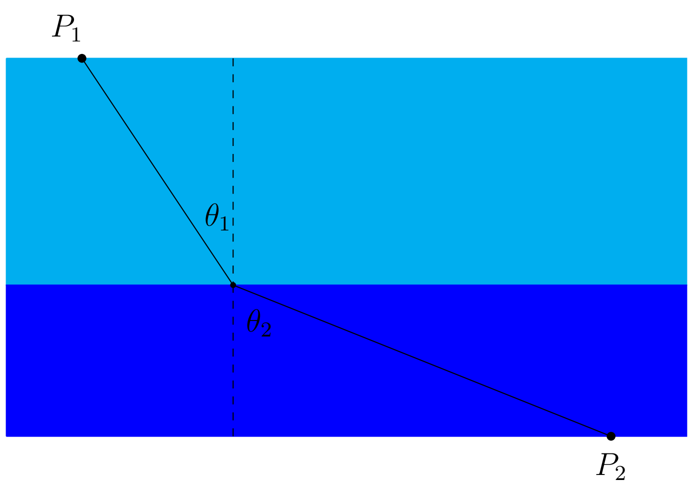
Math 1131 Applications: Optimization Fall 2019

We discuss here some extra optimization problems.

Application 1. Snell's law.

Light rays traveling from one medium to another, such as air to water, will bend. This causes items placed in water to look shifted, as in the images below.

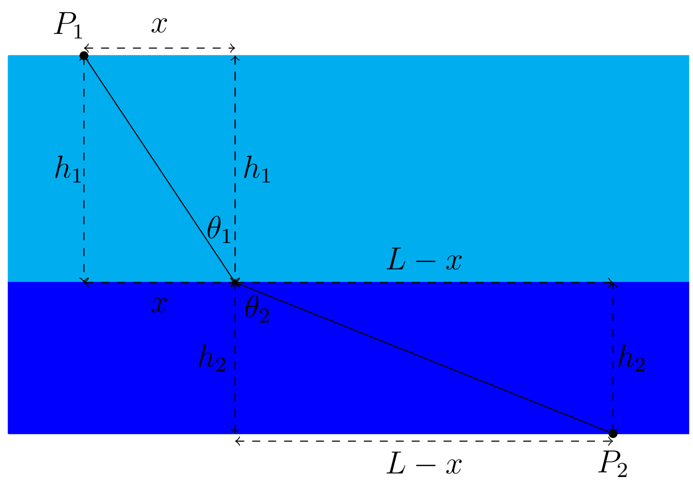
To describe the bending, a light ray from P_1 in air to P_2 in water (see below) takes a path through air at angle θ_1 and through water at angle θ_2 where $\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}}$, with v_1 being light's speed in air and v_2 its speed in water. This is Snell's law.



The physical basis for Snell's law is not that light takes the path from P_1 to P_2 of *least distance*, since that would be a straight line ($\theta_1 = \theta_2$), but instead that light follows a path of *least time*: the two straight lines along which a light ray travels from P_1 to P_2 are those for which the combined time along the paths is minimal. Let's use calculus to show the path of least time from P_1 to P_2 satisfies Snell's law.

We need a few more variables besides the angles, as in the figure below. Let

- x be the distance from P_1 to the line perpendicular to the boundary that passes through the point where a light ray from P_1 meets the boundary,
- h_1 be the vertical distance from P_1 to the boundary,
- h_2 be the vertical distance from P_2 to the boundary,
- L be the total horizontal separation between P_1 and P_2: $L = x + (L - x)$.



Let v_1 be the speed of light in air and v_2 be the speed of light in water. Let the light ray from P_1 to P_2 take time t_1 to travel from P_1 to the boundary and time t_2 to travel from the boundary to P_2. We can compute the distance from P_1 to the point where the light ray from P_1 reaches the boundary in two ways: by the distance formula with right triangles and by "rate times time":

$$\sqrt{x^2 + h_1^2} = v_1 t_1.$$

Using P_2 in place of P_1,

$$\sqrt{(L-x)^2 + h_2^2} = v_2 t_2.$$

Therefore the *total time* for the light ray to travel from P_1 to P_2 is

$$t_1 + t_2 = \frac{1}{v_1}\sqrt{x^2 + h_1^2} + \frac{1}{v_2}\sqrt{(L-x)^2 + h_2^2}.$$

We want an x in $[0, L]$ that minimizes the total time. The only flexible parameter is x, so write the total time above as a function of x, say $T(x)$. Its x-derivative is

$$T'(x) = \frac{1}{v_1}\frac{2x}{2\sqrt{x^2+h_1^2}} + \frac{1}{v_2}\frac{2(L-x)(-1)}{2\sqrt{(L-x)^2+h_2^2}} = \frac{1}{v_1}\frac{x}{\sqrt{x^2+h_1^2}} - \frac{1}{v_2}\frac{L-x}{\sqrt{(L-x)^2+h_2^2}},$$

so $T'(0) = -L/(v_2\sqrt{L^2+h_2^2}) < 0$ and $T'(L) = L/(v_1\sqrt{L^2+h_1^2}) > 0$. Since $T'(x)$ is continuous and positive at $x = 0$ and negative at $x = L$, we must have $T'(x) = 0$ for some x in $(0, L)$. In fact there is *only one* such x since T' is increasing on account of T' having a positive derivative: check with the quotient rule that for numbers $h > 0$,

$$\frac{d}{dx}\frac{x}{\sqrt{x^2+h^2}} = \frac{h^2}{(x^2+h^2)^{3/2}}, \quad \frac{d}{dx}\frac{L-x}{\sqrt{(L-x)^2+h^2}} = \frac{-h^2}{((L-x)^2+h^2)^{3/2}},$$

so

$$\begin{aligned} T''(x) &= \frac{1}{v_1}\frac{h_1^2}{(x^2+h_1^2)^{3/2}} - \frac{1}{v_2}\frac{-h_2^2}{((L-x)^2+h_2^2)^{3/2}} \\ &= \frac{1}{v_1}\frac{h_1^2}{(x^2+h_1^2)^{3/2}} + \frac{1}{v_2}\frac{h_2^2}{((L-x)^2+h_2^2)^{3/2}} \\ &> 0. \end{aligned}$$

At the critical number x in $(0, L)$ where $T'(x) = 0$ we have

$$\frac{1}{v_1}\frac{x}{\sqrt{x^2+h_1^2}} = \frac{1}{v_2}\frac{L-x}{\sqrt{(L-x)^2+h_2^2}}.$$

In terms of trigonometry, the left side is $(1/v_1)\sin\theta_1$ and the right side is $(1/v_2)\sin\theta_2$, so $(1/v_1)\sin\theta_1 = (1/v_2)\sin\theta_2$, which is the same as $(\sin\theta_1)/(\sin\theta_2) = v_1/v_2$. This path where $T'(x) = 0$ is a *minimum* of $T(x)$, not a maximum, by the second derivative test since $T''(x) > 0$. Thus a light ray from P_1 to P_2 that moves along a path of *least time* satisfies Snell's law $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$.

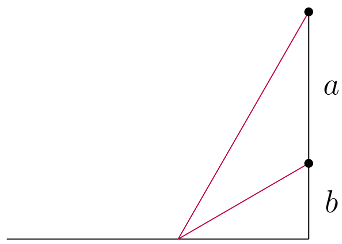
Snell's law (based on paths of least time) explains the appearance of a mirage on a hot road (with $\theta_2 = \pi/2$), sound waves across a lake and shock waves moving through different layers of rock.

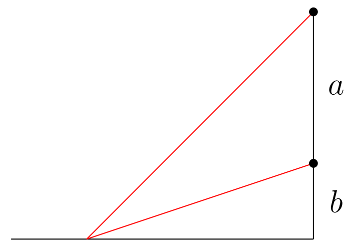
Application 2. Viewing angle maximization problem.

Suppose a painting (or movie screen) that is a feet tall is positioned vertically starting b feet from the ground, as shown below. At what distance from the wall, in terms of a and b, will the viewing angle from the ground to the screen be maximal?

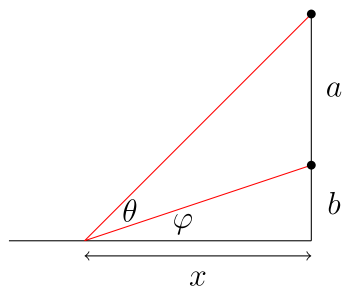
This problem is due to Regiomontanus in the 1400s. See here.

Since the viewer is *below* the lowest level of the screen, the viewing angle very close to the wall is small and the viewing angle very far from the wall is small, so the largest viewing angle is somewhere in between. Where is it?





At a distance x from the wall, let θ be the viewing angle (see figure below). If x is very small or very large then θ is very small, so there is some x where θ is maximal.



Rather than getting a formula for θ in terms of x (and a and b) we will get a formula for $\tan\theta$. At distance x from the wall let φ be the angle from the ground to height b on the wall, so by using right triangles, $\tan\varphi = b/x$ and $\tan(\theta+\varphi) = (a+b)/x$. We can get $\tan\theta$ from these using the addition formula $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$:

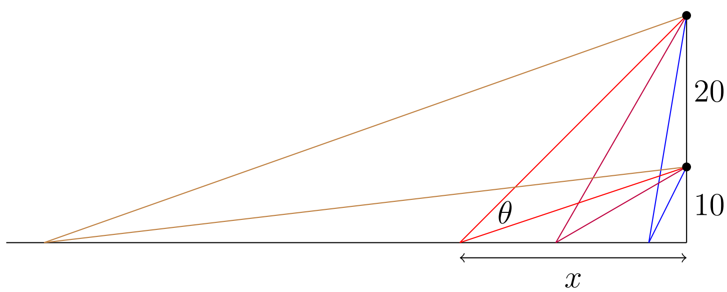
$$\begin{aligned}
\tan\theta &= \tan((\theta+\varphi)-\varphi) \\
&= \frac{\tan(\theta+\varphi)+\tan(-\varphi)}{1-\tan(\theta+\varphi)\tan(-\varphi)} \quad \text{using } \alpha = \theta+\varphi, \beta = -\varphi \\
&= \frac{\tan(\theta+\varphi)-\tan\varphi}{1+\tan(\theta+\varphi)\tan\varphi} \quad \text{since } \tan(-\varphi) = -\tan\varphi \\
&= \frac{(a+b)/x - b/x}{1+((a+b)/x)(b/x)} \\
&= \frac{ax}{x^2+(a+b)b}.
\end{aligned}$$

We want to maximize θ as x varies over positive numbers. Differentiate both sides with respect to x:

$$(\sec^2\theta)\frac{d\theta}{dx} = \frac{a(x^2+(a+b)b) - ax(2x)}{(x^2+(a+b)b)^2} = \frac{a((a+b)b - x^2)}{(x^2+(a+b)b)^2}.$$

The term $\sec^2\theta$ is positive. Since $x > 0$ we have $d\theta/dx = 0 \iff x = \sqrt{(a+b)b}$. If $0 < x < \sqrt{(a+b)b}$ then $d\theta/dx > 0$, and if $x > \sqrt{(a+b)b}$ then $d\theta/dx < 0$, so at $\boxed{x = \sqrt{(a+b)b}}$ the angle θ is maximal.

Example. A movie screen 20 feet high is placed on a wall starting at the 10 foot mark. At what distance from the wall, in feet, is the viewing angle towards the screen maximal?



From the work above, the maximal viewing angle occurs when $x = \sqrt{(20+30)10} = \sqrt{500} \approx 22.36$ feet.

Application 3. Optimization is not only something done to functions, but also to shapes. A Numberphile video about one such problem, called the "moving sofa problem" (see screenshot below) and still unsolved, is here. During 5:30-6:00 of the video, the speaker describes a shape being "locally optimal" and that is analogous to a local maximum or local minimum of a function. If you watch the whole video you'll find an unexpected connection to the movie Zoolander.