Math 1131 Applications: Exponential Growth/Decay Fall 2019

The most important use of derivatives in applications of calculus is the description of dynamically changing quantities by differential equations, which are equations involving an unknown function and its derivatives. Examples include $y'(t) = 3y(t)$ and $y''(t) = y(t)^2 - y(t)$. In applications, the variable t is usually time. People who care about solving a differential equation are interested in both approximations to a solution (with a computer) and qualitative features of a solution, *e.g.*, will a solution blow up in finite time? There is a million dollar prize for understanding the solutions to one particular differential equation.

The scope of applications of differential equations is vast:

- physics: gravitational, nuclear, and electromagnetic forces
- engineering: vibrations of mechanical systems, heat flow, electrical circuits
- chemistry: chemical concentrations during a reaction, molecular interactions
- biology: spread of infection, metabolism, population growth
- finance: movement of stock prices, pricing insurance products

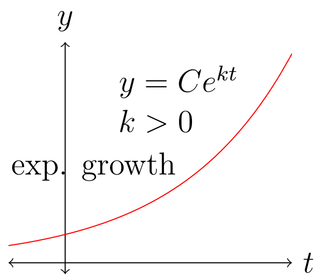
This just scratches the surface. A list of named differential equation is here.

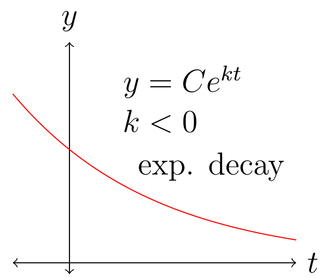
Remark. Computer simulation software for physical systems hides the underlying math, so the following remark taken from here is worth keeping in mind: "Perhaps the reason why some engineers and engineering students feel differential equations are not used by engineers is that they are working with simulating and modeling software [...] and don't see the actual mathematical model behind them."

The most basic widely applicable differential equation is $y'(t) = ky(t)$ for a constant k, and its general solution is

$$y(t) = Ce^{kt}$$

where $C = y(0)$ (the initial amount of y). The numbers C and k are constants, with C usually positive. If $k > 0$ these solutions describe exponential growth, and if $k < 0$ these solutions describe exponential decay. See the graphs below.





Examples of exponential growth include

- the size of a population with no predators or other factors restricting its size[1],
- the amount of money in an account subject to compound interest, particularly continuously compounded interest,
- a nuclear chain reaction.

Examples of exponential decay are

- the concentration of a drug in the blood after it is no longer being administered,
- atmospheric pressure as a function of height above sea level,
- the amount of remaining radioactive atoms in a pile.

A controlled use of exponential growth is how nuclear power plants work and is one reason nuclear bombs are hard to construct. This involves the proper handling of prompt and delayed neutrons. Improper handling is one cause of accidents at nuclear power plants.

Two Nobel prizes have been awarded to research involving physical quantities fitting the differential equation $y'(t) = ky(t)$:

1. 1960 Nobel in Chemistry to Libby for his creation of radiocarbon dating. This is a method of determining the age of old organic material such as prehistoric cave paintings and parchment manuscripts by measuring its carbon isotopes, one of which is subject to radioactive (exponential) decay.

[1]When there are predators, the population size both grows and decays. A basic model for this is the Lotka–Volterra equations.

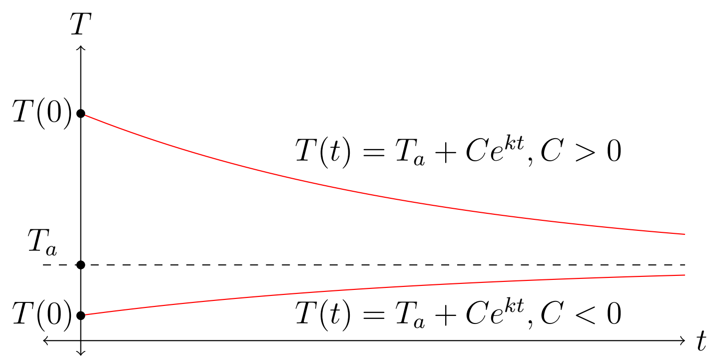
2. 1979 Nobel in Physiology or Medicine to Cormack and Hounsfield for their independent work on CT scanning. Cormack's mathematical work used the Beer–Lambert law, which comes from solving a differential equation that has $y'(t) = ky(t)$ as a special case.

One of the important parameters for a substance undergoing exponential decay is its half-life: the time needed for an amount of the substance to decay to half its value. For example, Gold-238 has a half-life of 2.7 days while Carbon-14 has a half-life of around 5730 years. To describe half-life from an exponential decay formula $y(t) = Ce^{kt}$ with $k < 0$, we want the time $t_{1/2}$ such that $y(t_{1/2}) = y(0)/2 = C/2$: $C/2 = Ce^{kt_{1/2}}$, so $e^{-kt_{1/2}} = 2$ (the factor $C = y(0)$ has canceled out). Taking natural logarithms of both sides, $-kt_{1/2} = \ln 2$, so $t_{1/2} = (\ln 2)/(-k) = (\ln 2)/|k|$.

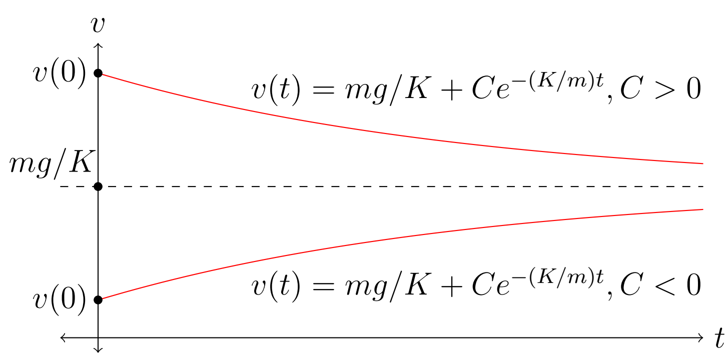
Some differential equations that are not directly of the form $y'(t) = ky(t)$ for constant k can be solved in a similar way to that equation, such as $\boxed{y'(t) = k(y(t) - b)}$ for constants k and b. This implies $(y(t) - b)' = k(y(t) - b)$, so $y(t) - b = Ce^{kt}$ for a constant C, and thus $\boxed{y(t) = b + Ce^{kt}}$. (Here C is *not* $y(0)$, but $y(0) - b$.) If $k < 0$ then $y(t) \to b$ as $t \to \infty$, so b is the "terminal" (limiting) value of $y(t)$ for large t. Think of a hot object cooling down to room temperature, a cool object warming up to room temperature, or a falling object reaching terminal velocity. Here are examples of this differential equation in action.

- Newton's law of cooling for an object placed in a large room says its temperature decays at a rate proportional to the difference between its current temperature and the ambient (room) temperature: $T'(t) = k(T(t) - T_a)$, where $k < 0$ and T_a is the ambient temperature. This matches the boxed differential equation above, with $b = T_a$. Then $T(t) = T_a + Ce^{kt}$, so from $k < 0$ we get $T(t) \to T_a$ as $t \to \infty$, which says the object's temperature approaches room temperature, a familiar physical result. If $T(0) > T_a$ then we have cooling, while if $T(0) < T_a$ then we have warming. See the graph below. Newton's law of cooling is a good approximation when the object's initial temperature $T(0)$ is within 50° F of T_a. (The validity when $|T(0) - T_a|$ is small is like the approximation $\sin\theta \approx \theta$ only being good when θ is small in radians.)

- A model for an object moving through air subject to air drag: if air drag is proportional to the velocity, which is a good approximation for falling mist particles of oil or water, then from Newton's second law the object's velocity $v(t)$ satisfies the differential equation $mv'(t) = mg - Kv(t)$ where m is the mass, $g \approx 9.8\text{m/s}^2$ and K is a positive constant depending on physical properties of the object and air. The equation is the same as $v'(t) = -(K/m)(v(t) - mg/K)$, which matches the boxed differential equation above ($k = -K/m$ and $b = mg/K$), so $v(t) = mg/K + Ce^{-(K/m)t}$ for some C. The terminal velocity (limit of $v(t)$ as $t \to \infty$) is mg/K. See the graph below. This was used by Millikan in his oil drop experiment, which was the first measurement of the electron charge and used the terminal velocity of falling oil drops. It earned Millikan the 1923 Nobel prize in physics.



Temperature subject to Newton's Law of Cooling



Velocity when air drag is proportional to velocity

(For a falling skydiver, drag is proportional not to the velocity but to the *square* of the velocity, so $mv'(t) = mg - Kv(t)^2$ for some positive constant K, and thus $v'(t) = -(K/m)(v(t)^2 - mg/K)$ after some algebra. Solving this differential equation uses methods from Math 1132, so we don't work it out here. We'll just say there is a terminal velocity again: now it's $\sqrt{mg/K}$ instead of mg/K.)