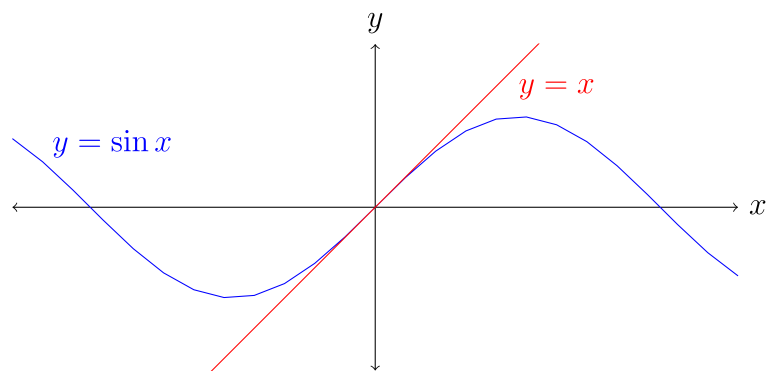
Math 1131 Applications: Small-Angle Approximation Fall 2019

That $\sin'(0) = \cos(0) = 1$ means the tangent line to the graph of $y = \sin x$ at $x = 0$ has slope 1: the tangent line is $y = x$. See the picture below, where the graph of $y = \sin x$ for x near 0 is approximated well by the graph of $y = x$.



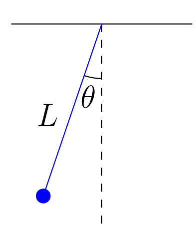
That $\boxed{\sin x \approx x}$ for small x is called a *small-angle approximation*. It is illustrated numerically in the table below. The angles are in radians, so $.2 = .2$ radians $\approx 11.4°$ (multiply by $180/\pi$ to convert from radians to degrees).

x	.2	.1	.023	.00452	.00059	.000328
$\sin x$	.198669	.099833	.022997	.004519	.000589	.0003279

Continuity of $\sin x$ at $x = 0$ tells us $\sin x \to \sin 0 = 0$ as $x \to 0$. The small-angle approximation for $\sin x$, which is based on differentiability, is an improvement on what we learn from continuity: the small-angle approximation tells us *how* $\sin x$ tends to 0 as $x \to 0$: in a linear (first-power) way. Being able to replace the complicated function $\sin x$ with the function x, when x is small, is a convenient approximation in applications.

Application 1. Small oscillations of a pendulum.

If we set a small pendulum in motion, it oscillates back and forth as shown below.



If the pendulum is released with velocity 0, its displacement angle $\theta = \theta(t)$ from a vertical position varies with time, and an equation describing $\theta(t)$ turns out to be (using Newton's second law and ignoring friction and air drag)

$$\theta''(t) + \frac{g}{L}\sin\theta(t) = 0 \text{ with } \theta'(0) = 0,$$

where L is the length of the pendulum and $g \approx 9.8$ m/s^2 is the acceleration due to gravity near the surface of the earth. The above equation is analytically hard to solve for $\theta(t)$, but when $\theta(t)$ is small (in radians, so $10^\circ \approx .174$ radians is small) we can approximate the term $\sin\theta(t)$ by $\theta(t)$, which leads to the equation

$$\theta''(t) + \frac{g}{L}\theta(t) = 0 \text{ with } \theta'(0) = 0,$$

and this *can* be solved: $\boxed{\theta(t) = \theta(0)\cos(\sqrt{g/L}t)}$, where $\theta(0)$ is the initial (release) angle for the pendulum at time $t = 0$. (Note $\theta'(t) = -\theta(0)\sin(\sqrt{g/L}t)\sqrt{g/L}$, so $\theta'(0) = 0$, which corresponds to the initial release velocity being 0.) Here are two interesting observations about the formula for $\theta(t)$:

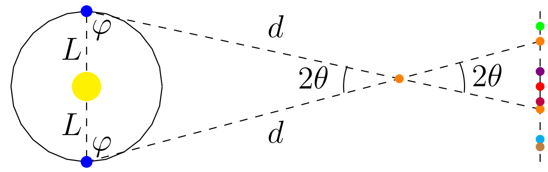
1. Since $\cos x$ has values in $[-1, 1]$, the displacement angle $\theta(0)\cos(\sqrt{g/L}t)$ has values in $[-\theta(0), \theta(0)]$: this means the pendulum returns to its original release angle but not a higher one. See this with a bowling ball pendulum here.

2. The period T of $\theta(0)\cos(\sqrt{g/L}t)$ as t varies is[1] $2\pi\sqrt{L/g}$, which is independent of the release angle $\theta(0)$. So the period of a pendulum with different *small* release angles have the same period T. See this shown for a few different angles here. That T does not depend on $\theta(0)$ when $\theta(0)$ is small is the basis for pendulum clocks, which were the primary timekeeping mechanism for over 250 years. If $\theta(0)$ is not small, T does depend on it: such formulas are here, which are expansions in infinite series (a topic in Math 1132) having $2\pi\sqrt{L/g}$ as the first term.

Application 2. Measuring the distance to stars.

The approximation $\sin\theta \approx \theta$ for small θ is the basis for the parallax method of estimating the distance between the Earth and other stars (except the Sun). See the diagram below.

[1]For $A > 0$ and $c > 0$, the period of $A\cos(c\theta)$ is $2\pi/c$.



Stars except the Sun are so far away that they largely don't appear to move at all relative to each other. This is why ancient constellations look largely the same today and why ancient astronomers referred to a background of "fixed stars" against which the planets move (the word "planet" is from the Greek term for wanderer). For some stars, it was possible by the 1800s to detect a *small* apparent motion relative to the background of "fixed" stars when observed at different times: the apparent positions of the same star 6 months apart (meaning the Earth is on opposite sides of the Sun) sweeps out a very small angle. An analogy you can check in your room is viewing your finger in front of you with just one eye and then just the other eye; your finger has not physically moved, but it will appear to have moved against the background wall (or window, *etc.*).

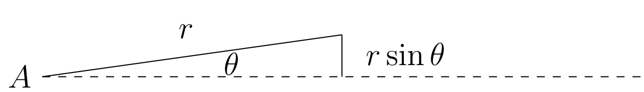
Let 2θ be the angle a star appears to sweep out over 6 months (this angle is called the *parallax* of the star). Since this angle is so small, the triangle connecting the star to the positions of the Earth 6 months apart has two sides of essentially equal length d, so we treat the triangle as isoceles where the two equal angles φ are nearly $90°$. Using the Law of sines, $\sin(2\theta)/(2L) = \sin(\varphi)/d \approx 1/d$. Since 2θ is very small we can say $\sin(2\theta) \approx 2\theta$, so

$$\frac{2\theta}{2L} \approx \frac{1}{d} \Longrightarrow d \approx \frac{L}{\theta}.$$

This is how the distance d to the star is measured.

Application 3. Pilot navigation

If a pilot intends to fly along a certain straight line route but is off from that direction by a small angle θ, the "1 in 60" rule says that each $1°$ error in direction leads to a 1 mile error from the planned flight path (1 mile "off track") for every 60 miles flown. We'll explain this with the diagram below.



If you want to fly from A due east but travel instead at a small nonzero angle

θ from an eastern direction, then after traveling r miles the (straight line) distance the plane is from the intended direction is $r \sin \theta$, which for small θ (in radians!) is around $r\theta$. Since $1^\circ = \pi/180$ radians, and $\pi/180 \approx 3/180 = 1/60$, we obtain for $r = 60$ miles and $\theta = 1^\circ$ that $r\theta = 60\theta \approx 1$ mile. (The actual distance "off track" is $60 \sin(\pi/180) \approx 1.047$ miles.) Some examples aimed at an audience of pilots is here.