Math 1131Q Week 2: Section 2.4 Guided Activity

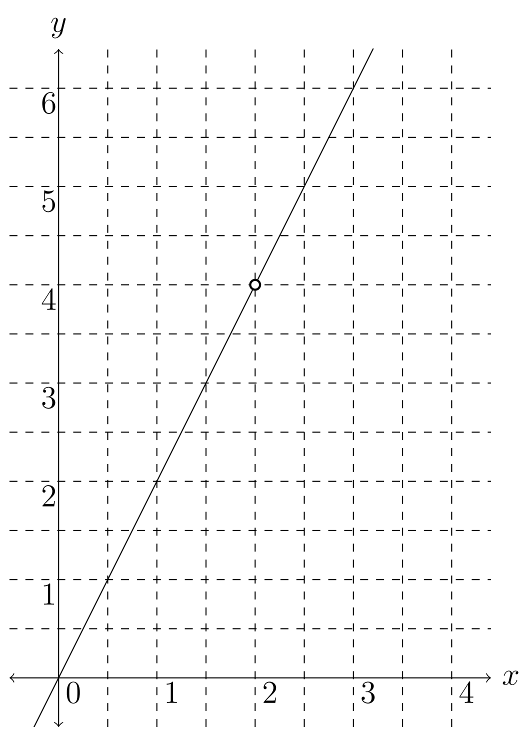
Name: ____________________________________

Discussion Section: _________________________

2.4: The Precise Definition of a Limit

The central idea in calculus is limits: how a function behaves when its input "tends to" a specific value. In Sections 2.1–2.3, we saw a setting where limits are needed to determine instantaneous rates of change (the slope of a tangent line is the limit of slopes of secant lines) and we saw limits of functions behave nicely for some operations (like adding and multiplying functions). While calculus was developed for over 100 years without limits, eventually the importance of the concept was recognized and it received a precise mathematical definition. We want you to have some experience with this definition, and that is the purpose of these guided notes. *The material is challenging, and will not be used later on.*

Below is the graph of the function $f(x) = \dfrac{2x(x-2)}{x-2}$, which is undefined at $x = 2$ (i.e., the graph is that of $y = 2x$ with a hole in the graph at the point $(2, 4)$). Visually we can see $\lim_{x\to 2} f(x) = 4$: as x gets "closer and closer" to 2, $f(x)$ gets "closer and closer" to 4. How do we give each "closer and closer" a rigorous meaning together?



Suppose we want to stay on the graph so that the y-values differ from 4 by at most 1. To achieve that, how close should x stay to 2?

How close should x stay to 2 if, on the graph, we want y to differ from 4 by at most 1/2?

How close should x stay to 2 if, on the graph, we want y to differ from 4 by at most some value h?

Describe how your previous answer would change if we consider behavior near $x = 2$ for the function $g(x) = \dfrac{mx(x-2)}{x-2}$, where $m > 0$ is a constant: how close should x stay to 2 if we want $g(x)$ to differ from $2m$ by at most h?

This example used $x = 2$, but x can have other values. When we say that the limit of a function $f(x)$, as x approaches a, is L (written $\lim_{x \to a} f(x) = L$), this means that if we want $f(x)$ to differ from L by some chosen amount (which is usually denoted by the Greek letter ____________, written as ___), then we can make this happen when x is close enough to a.

The variable traditionally used to denote how close x being to a is "close enough" to make this work is the Greek letter ____________, written as ___. In other words, "when x is close enough to a" is written as "when $0 < |x - a| <$ ___" (We have $0 < |x - a|$ to avoid using $x = a$: we don't assume $f(x)$ is necessarily defined at $x = a$ when we speak of its limit at a).

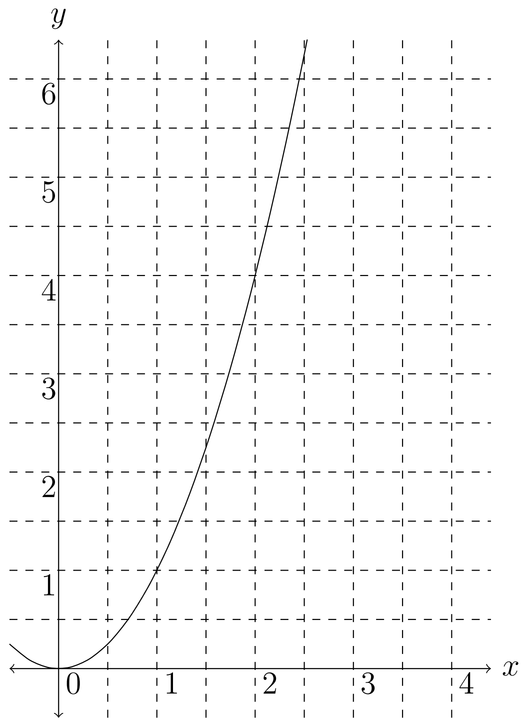
Formally, what we mean when we write $\lim_{x \to a} f(x) = L$, is the following:

> **Epsilon-Delta Definition of Limit:** For every $\varepsilon > 0$, there is a value $\delta > 0$ such that whenever $0 < |x - a| <$ ___, it follows that $|f(x) - L| <$ ___.

In words, we say how close we want $f(x)$ to be to L *first*, where close is measured by $|f(x) - L|$ being less than some some number ε, and no matter how close that is ("for every $\varepsilon > 0 \ldots$") we can achieve it for all x that are __________________ to a, excluding $x = a$ itself. That closeness of x to a is measured by δ. (If there is one or more ε where we *can't* do this, then $\lim_{x \to a} f(x)$ is *not* L.)

In the above example with the graph of a line with a hole at $(2, 4)$, note that we can always get $f(x)$ as close to 4 as we want by having x be ____________________________________ to (but not equal to ___). It never matters if $f(x)$ is actually defined *at* ___. This is because when we write $0 < |x - 2| < \delta$, the part ____________ "ignores" what happens at $x = 2$.

Here is another example. Below is the graph of $y = f(x)$ where $f(x) =$ ______. The limit of $f(x)$ as $x \to 2$ is ______.



How close (what's δ?) should we be to $x = 2$ if we want $f(x)$ to be within $\varepsilon = 1$ of ______? Draw the lines $y =$ ______ and $y =$ ______ from where they cross the y-axis to where they meet the graph and then draw vertical lines down to the x-axis. The points you get on the x-axis are on both sides of $x = 2$, but are *not* the same distance from 2.

Right distance to 2: _______________ Left distance to 2: _______________

The right and left distances from $x = 2$ are different. Is it good to let δ be the larger or smaller of the distances, and why?

$\delta =$ ___________________ distance to $x = 2$ from right and left, which is ____________,

because ____.

Let's try another ε. How close (what's δ?) should x be to 2 if we want $f(x)$ to be within $\varepsilon = .5$ of ______? Draw the lines $y =$ ____________ and $y =$ ____________ until they meet the graph and then draw vertical lines to the x-axis on each side of $x = 2$.

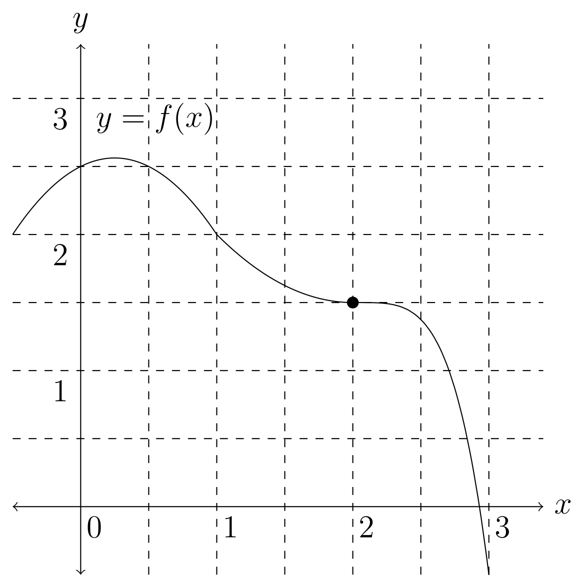
Right distance to 2: _______________ Left distance to 2: _______________

$\delta =$ ____________

Exercises.

1. For the continuous function $f(x)$ whose graph is below, $f(2) = 1.5$.

   Estimate a value of $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|f(x) - 1.5| < 1/2$. Explain your answers by referencing the graph.



2. Let $f(x) = 4x - 1$. We will show $\lim_{x \to 2} f(x) = 7$ with the ε-δ definition of a limit when $a = 2$ and $L = 7$. That is, we aim to prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| =$ ______ $< \delta$ it follows that $|f(x) - L| =$ ____________ $< \varepsilon$.

   (a) For $\varepsilon = 0.1$, find δ so that whenever $0 < |x - 2| < \delta$ we get $|f(x) - 7| < \varepsilon$.

   (b) For $\varepsilon = 0.01$, find δ so that whenever $0 < |x - 2| < \delta$ we get $|f(x) - 7| < \varepsilon$.

   (c) For $\varepsilon > 0$, find δ (in terms of ε) so that whenever $0 < |x - 2| < \delta$ we get $|f(x) - 7| < \varepsilon$.

Using this process, we can now say that for every $\varepsilon > 0$ there is a $\delta > 0$, namely

$\delta =$ ______, such that whenever $0 < |x -$ ______$| < \delta =$ ____________ we get

$|f(x) - L| =$ ___________________ $=$ ____________ $=$ _________ $<$ ______ $= \varepsilon$.

Thus we have used the ε-δ definition of limits to show ___________________.

(Using the ε-δ definition for limits of functions that are not linear is harder because δ depends in a more complicated way on a and $f(x)$.)