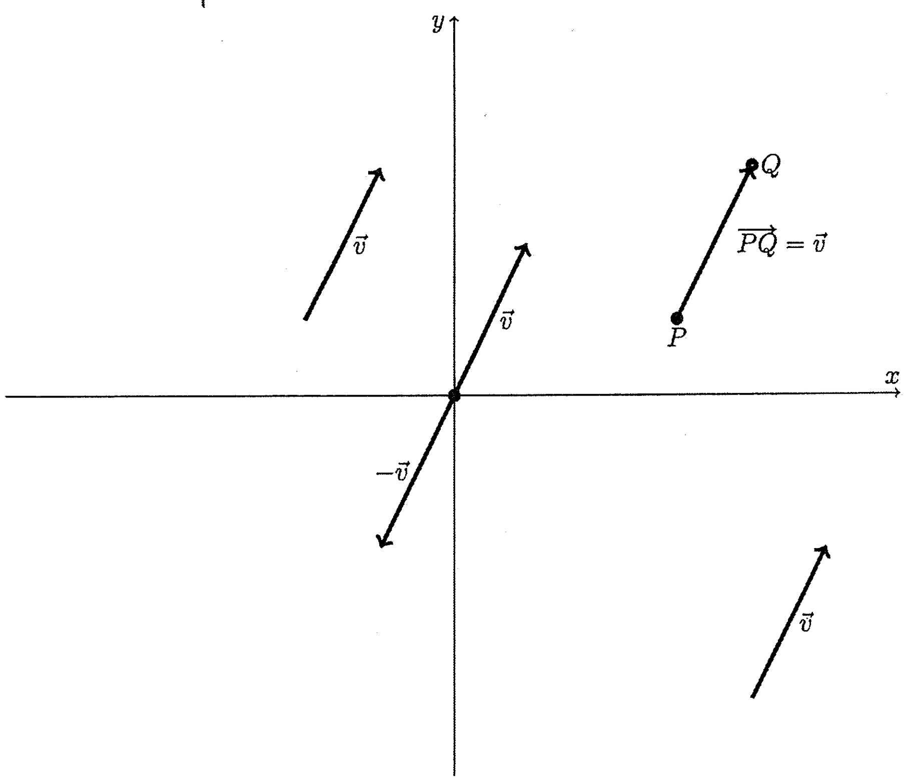
§ 12.2 Vectors

What is a vector?

A **vector** is an object with both magnitude and direction, whereas a **scalar** has only Magnitude.

Some examples of a scalar quantity are speed, energy, voltage, and distance, but we would use a vector to describe quantities like velocity, force, torque, and displacement, for example.

A vector $\vec{v}$ is the same no matter where it starts and ends, so long as the magnitude and direction are equal. We typically call a vector a position vector if it starts at the origin.



We often write a vector with an arrow over a letter to distinguish a vector $\vec{v}$ from a scalar v, but we also can talk about the vector between two points P and Q, denoted $\overrightarrow{PQ}$.

We may refer to a vector $\vec{v}$ by its components as well, writing $\vec{v} = \langle x, y, z\rangle$. Using this notation will be helpful with computations and gives us another way to see how operations work with vectors. Also, a typical notation (especially in engineering) is to use $\hat{\imath} = \langle 1,0,0\rangle$, $\hat{\jmath} = \langle 0,1,0\rangle$, and $\hat{\mathbf{k}} = \langle 0,0,1\rangle$. Using these conventions, we can write any vector $\vec{v} = \langle x,y,z\rangle$ as $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$.

We can think of a point (x, y, z) as the position vector $\langle x, y, z\rangle$, which connects the origin to the point (x, y, z). This will be helpful in future applications of vectors.

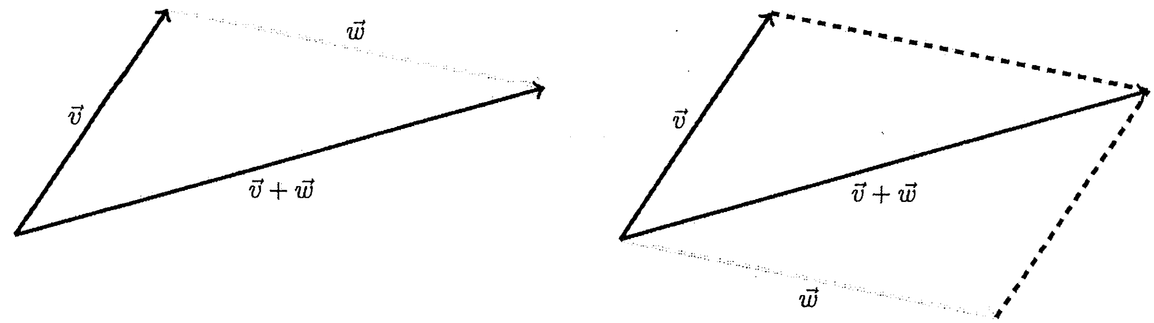
The zero vector is a special vector has no magnitude and points in every direction, denoted by

$$\vec{0} = \langle 0,0,0\rangle.$$

Note that for any vector $\vec{v}$, we have $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$.

Addition of Vectors

We can add any two vectors either graphically or using their components. The following picture shows the two main methods for adding vectors graphically: the Tip-to-Tail Method and the Parallelogram Method:



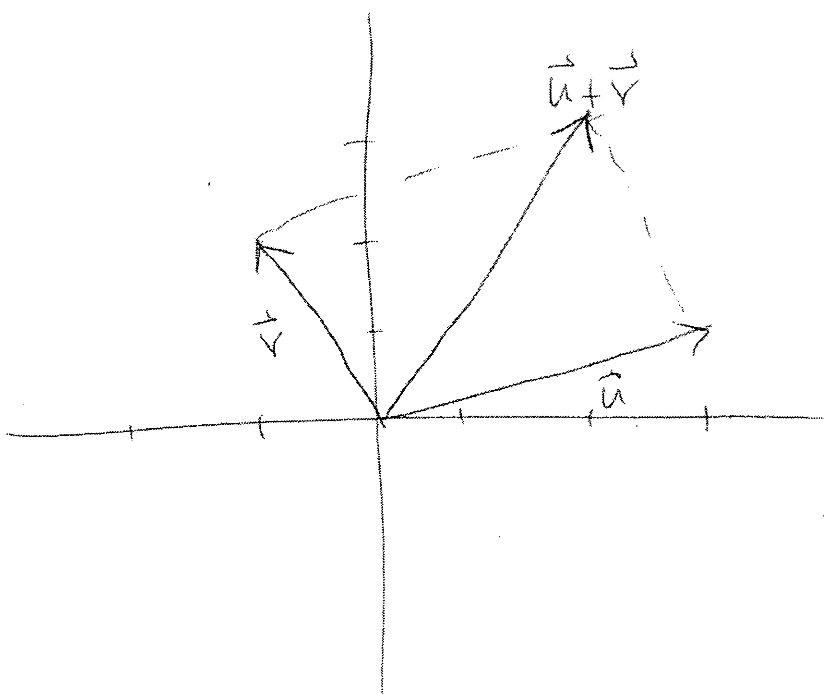
We can also add two vectors using their component form. Say $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$. Then

$$\vec{a}+\vec{b} = \langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle$$

For example, $\langle 1,-2,1\rangle + \langle 2,0,5\rangle = \langle 3,-2,6\rangle$.

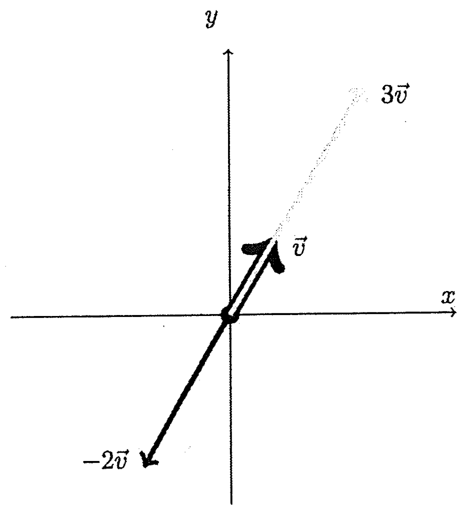
Example 1: Find the sum of the vectors $\vec{u} = \langle 3,1\rangle$ and $\vec{v} = \langle -1,2\rangle$ both graphically and by adding the components.

$\vec{u}+\vec{v} = \langle 3,1\rangle + \langle -1,2\rangle = \langle 2,3\rangle$



Scalar Multiplication

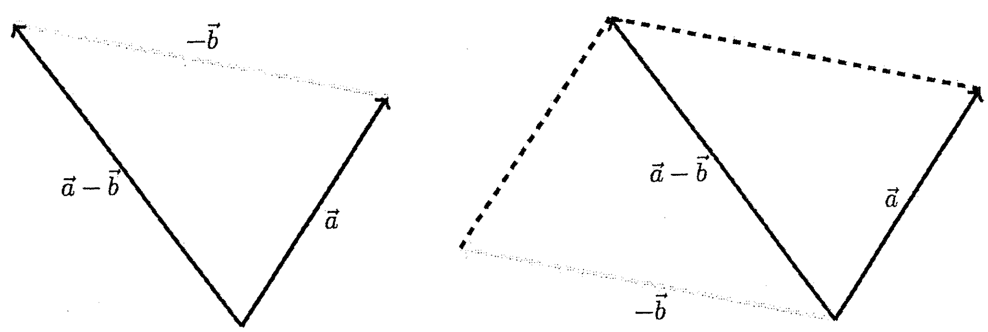
We can multiply a vector $\vec{v} = \langle x, y, z \rangle$ by any scalar c by $c\vec{v} = \langle cx, cy, cz \rangle$. If $c > 0$, the vector $c\vec{v}$ will point in the same direction as $\vec{v}$, but it will point in the opposite direction of $\vec{v}$ if $c < 0$.



Subtraction of Vectors

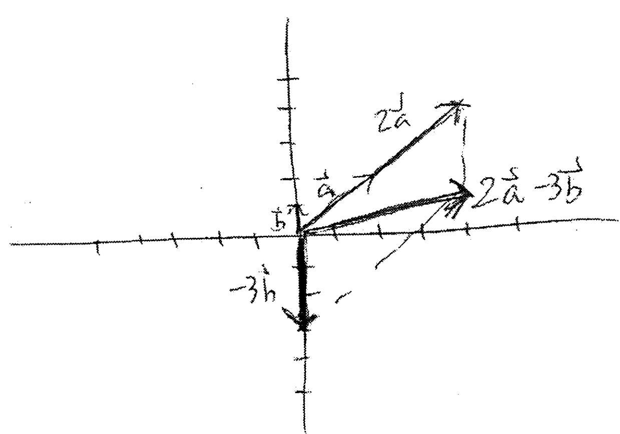
Now that we have defined addition and scalar multiplication of vectors, we can define subtraction. We define subtraction via

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$



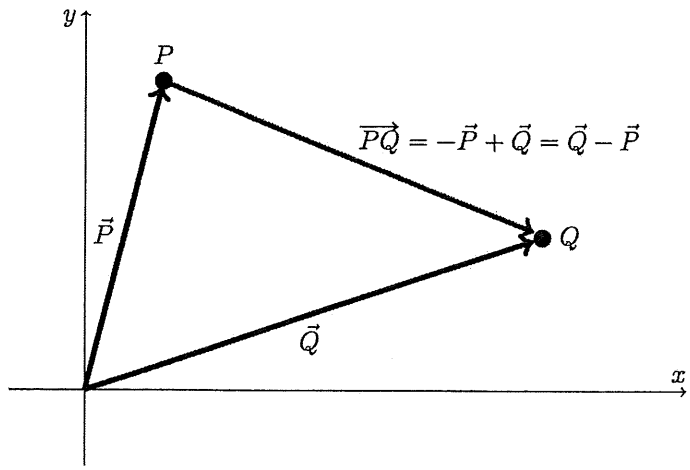
Example 2: If $\vec{a} = \langle 2, 2 \rangle$ and $\vec{b} = \langle 0, 1 \rangle$, find $2\vec{a} - 3\vec{b}$. Sketch this sum graphically as well.

$2\vec{a} - 3\vec{b} = 2\langle 2,2\rangle - 3\langle 0,1\rangle = \langle 4,4\rangle - \langle 0,3\rangle = \langle 4,1\rangle$



Vectors Between Points

Given two points P and Q, we often want to find the vector from P to Q, denoted $\overrightarrow{PQ}$.



If we know the components of the points involved, then we can compute the components of the vector $\overrightarrow{PQ}$. For example, if we want the vector from $P(1,3,-1)$ to $Q(-2,0,3)$, then we compute

$$\overrightarrow{PQ} = \vec{Q} - \vec{P} = \langle -2-1, 0-3, 3-(-1)\rangle = \langle -3,-3,4\rangle.$$

Example 3: Find the vector that starts at the point $P(0,3,-5)$ and ends at $Q(-2,-1,3)$.

$$\overrightarrow{PQ} = \langle -2-0, -1-3, 3-(-5)\rangle = \langle -2, -4, 8\rangle$$

Magnitude of a Vector

For a vector $\vec{v}$, we denote its magnitude (length) by $|\vec{v}|$. If $\vec{v} = \langle x,y,z\rangle$, then

$$|\vec{v}| = \sqrt{x^2+y^2+z^2},$$

which is the distance from the origin to the point (x,y,z).

Example 4: Find the lengths of the vectors $\vec{a} = \langle 1,4,-2\rangle$, $\vec{b} = \langle 2,-3,-1\rangle$, and $\vec{a}+\vec{b}$.

$|\vec{a}| = \sqrt{1+16+4} = \sqrt{21}$, $|\vec{b}| = \sqrt{4+9+1} = \sqrt{14}$

$\vec{a}+\vec{b} = \langle 3,1,-3\rangle$, so $|\vec{a}+\vec{b}| = \sqrt{9+1+9} = \sqrt{19}$

We say that a vector $\vec{u}$ is a <u>unit vector</u> if it has length 1, that is, $|\vec{u}| = 1$. For any vector $\vec{v}$, we can always find a unit vector in the same direction by taking

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}.$$

Unit vectors are important in many applications. Note that if $\vec{u} = \langle x, y \rangle$ is a unit vector, then the point (x, y) lies on the unit circle $x^2 + y^2 = 1$.

<u>Example 5:</u> Given the vector $\vec{v} = \langle 1, -3, 4 \rangle$, find a unit length vector with the same direction as $\vec{v}$ and another that has the opposite direction.

$|\vec{v}| = \sqrt{1+9+16} = \sqrt{26}$, so

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{26}}\langle 1, -3, 4 \rangle \overset{\text{or}}{=} \left\langle \frac{1}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right\rangle$$

Another unit vector in the opposite direction is

$$-\vec{u} = \frac{-1}{\sqrt{26}}\langle 1, -3, 4 \rangle \overset{\text{or}}{=} \left\langle \frac{-1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{-4}{\sqrt{26}} \right\rangle$$