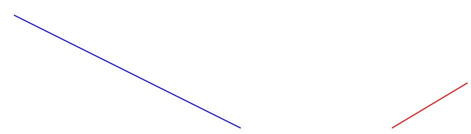
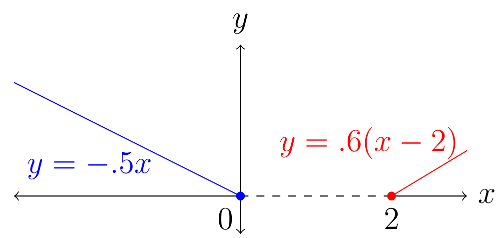
Math 1131 Applications: Curve fitting **Fall 2019**

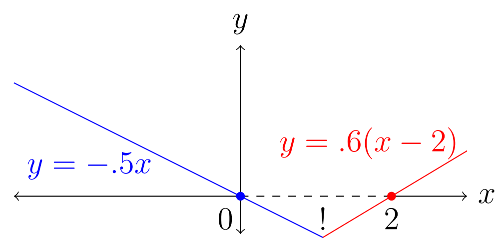
Suppose we have the two straight parts of a roller coaster track as shown below.



How can we find a curve connecting them that will give passengers a smooth ride as they transition from one straight track to the other? First let's introduce coordinates so we can describe things with equations, say the following.



We certainly do not want the transition track to be the straight line extensions of each track until they meet. Imagine what happens when the roller coaster reaches the corner!



To get a good transition curve we will find a *cubic polynomial* that works well. (Why not quadratic? We'll explain that later) Set

$$f(x) = \begin{cases} -.5x, & \text{if } x \le 0, \\ ax^3 + bx^2 + cx + d, & \text{if } 0 \le x \le 2, \\ .6(x-2), & \text{if } x \ge 2 \end{cases}$$

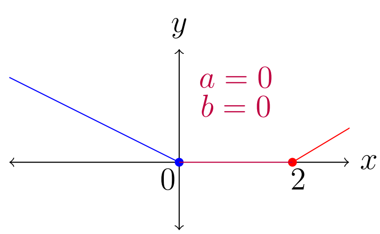
for some coefficients a, b, c, d that we need to find. First of all we want the whole track to be **continuous**: since $f(0) = 0$ and $f(2) = 0$, we need

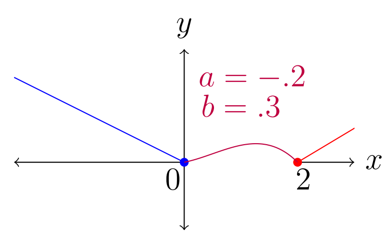
$$\lim_{x\to 0^+} f(x) = 0 \Rightarrow d = 0, \quad \lim_{x\to 2^-} f(x) = 0 \Rightarrow 8a + 4b + 2c = 0 \Rightarrow c = -(4a + 2b).$$

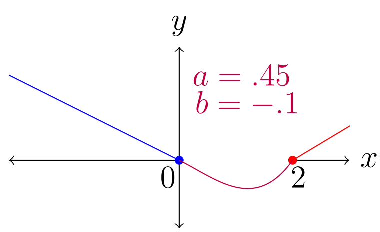
Then $ax^3 + bx^2 + cx + d = ax^3 + bx^2 - (4a + 2b)x = a(x^3 - 4x) + b(x^2 - 2x)$, so

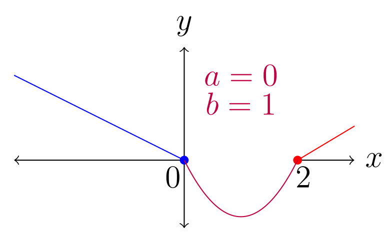
$$f(x) = \begin{cases} -.5x, & \text{if } x \le 0, \\ a(x^3 - 4x) + b(x^2 - 2x), & \text{if } 0 \le x \le 2, \\ .6(x - 2), & \text{if } x \ge 2. \end{cases}$$

We need to find good choices for a and b. Most choices lead to a bad track:









To make the transition curve smoothly match the lines, we want *differentiability* at the two transition points. Derivatives away from $x = 0$ and $x = 2$ are

$$f'(x) = \begin{cases} -.5, & \text{if } x < 0, \\ a(3x^2 - 4) + b(2x - 2), & \text{if } 0 < x < 2, \\ .6, & \text{if } x > 2, \end{cases}$$

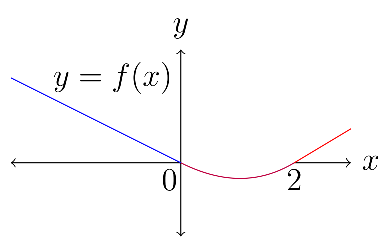
and taking limits as $x \to 0^{\pm}$ and $x \to 2^{\pm}$ tells us we want

$$\begin{aligned} -.5 &= -4a - 2b \\ .6 &= 8a + 2b. \end{aligned}$$

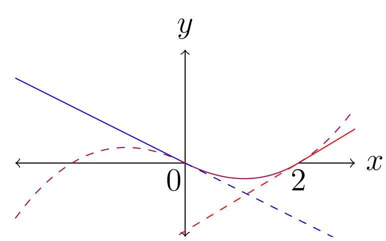
This is two equations in two unknowns, and after some algebra we get the solution $a = .025$ and $b = .2$:

$$f(x) = \begin{cases} -.5x, & \text{if } x \le 0, \\ .025(x^3 - 4x) + .2(x^2 - 2x), & \text{if } 0 \le x \le 2, \\ .6(x-2), & \text{if } x \ge 2. \end{cases}$$

The left figure below shows how the resulting track appears, and it looks good. The right figure below is the graph for all x of all three formulas making up $f(x)$. At both $(0, 0)$ and $(2, 0)$ the graphs have matching tangent lines from the left and right, which makes the roller coaster track look smooth.



The finished track!



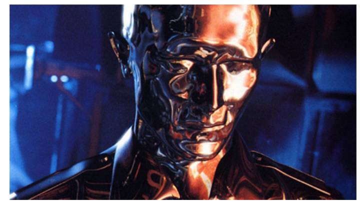
Graphs of all three functions

Now we can explain why we used a cubic polynomial for the transition curve. There were four conditions we needed the final curve to satisfy: continuity at each transition point and differentiability at each transition point. A quadratic polynomial $ax^2 + bx + c$ only has 3 coefficients, which is not enough flexibility when we want the final curve to satisfy 4 constraints.

We only discussed matching function values and first derivatives on a transition curve, but for designs in the real world second derivatives also matter thanks to Newton's second law, which expresses force in terms of a second derivative ($F = ma = mx''(t)$). Not paying attention to second derivatives might lead to uncomfortable forces being felt at the transition even if it looks smooth. This matters not only with transition curves for roller coasters, but also for railroad tracks.

Aside from lines transitioning into curves, we may want curves to transition into curves, This is often done with splines, which is a method of curve and surface fitting used in car design and computer graphics, as indicated below (the second image is taken from Terminator 2).

Car design



Computer Graphics

Before computers made spline calculations feasible, automotive designers created transition curves by hand, using a tool with many different curves on it called a French curve, shown below on the left. The image on the right shows two automotive engineers in 1930 and their large supply of French curves.

Another application of curve fitting is the creation of scalable fonts, like the "S" below at two different sizes.

Ss

The next time you zoom in on a web page and the letters on the screen doesn't pixelate on you, calculus had a role in that.