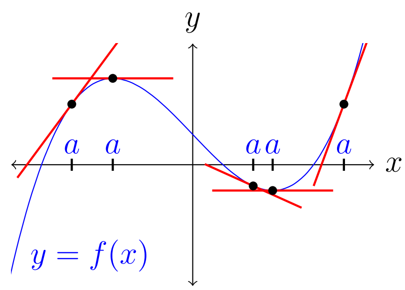
Math 1131 Applications: Derivatives **Fall 2019**

The derivative of a function $f(x)$ at the number a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Geometrically this is the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$, and this slope usually depends on a as in the graph below.



The versatility of the derivative is due to its many other interpretations.

- Rate of change: for $x \neq a$, the ratio $(f(x) - f(a))/(x - a)$ is the rate of change of the function over the interval between x and a. Letting $x \to a$ then gives the interpretation of $f'(a)$ as the *instantaneous* rate of change of $f(x)$ at $x = a$ (in applications, x is usually time and written as t). There are many quantities we care about in the world that are dynamic so we care about how they are changing, and this is measured by their rate of change: position (rate of change is velocity), electric charge (rate of change is current), population (rate of change is population growth rate), heat, fluid flow, chemical reactions, drug dosage level in the bloodstream, stock prices, and so on. A 3Blue1Brown video about the instantaneous rate of change is here.

- Relative rate of change: in biology and economics, it might be $f'(x)/f(x)$ that is of greater interest than $f'(x)$ alone. We call $f'(x)/f(x)$ the *relative* rate of change (as a fraction of the current value), although in those other disciplines they might use the term "rate of change" for that ratio.

- Linear approximation: for x near a, $f'(a) \approx \dfrac{f(x) - f(a)}{x - a}$, so $f(x) - f(a) \approx f'(a)(x - a)$, so $\boxed{f(x) \approx f(a) + f'(a)(x - a)}$. So $f(a) + f'(a)(x - a)$ is a *linear* *polynomial* in x and it approximates $f(x)$ when $x \approx a$. (It's exactly the expression for the tangent line to the graph of $f(x)$ at $(a, f(a))$: $y = f(a)+f'(a)(x-a)$.)

For example, if $f(x) = x^2$ then $f'(a) = 2a$, so in particular $f'(3) = 6$: for $x \approx 3$, $x^2 \approx 9+6(x-3)$. We can see this numerically: if $x = 3.001$ then $x^2 = 9.006001$ and $9 + 6(x - 3) = 9 + 6(.001) = 9.006$.

- Error propagation: for x near a, the formula $\boxed{f(x) - f(a) \approx f'(a)(x - a)}$ says that the function $f(x)$ approximately scales errors $x-a$ by the numerical factor $f'(a)$, so derivatives help estimate *how much* errors are magnified or shrunk when a function is applied.

  For example, if $f(x) = x^2$, so $f'(3) = 6$, when $x \approx 3$ we have $x^2 - 9 \approx 6(x - 3)$. Checking this for $x = 3.001$, $x^2 - 9 = 0.006001$ and $6(x - 3) = 6(.001) = 0.006$.

- While the derivative is a limit as $h \to 0$, in applications to economics the smallest positive value of h in many situations might be 1 (*e.g.*, if h is the number of units of some good being sold). Writing $R(x)$ for the revenue function (how much revenue is obtained from selling x units), $(R(x + h) - R(x))/h$ at $h = 1$ is $R(x + 1) - R(x)$, so in economics $R'(x)$ is a convenient approximation for $R(x+1) - R(x)$, whose meaning is: the revenue from selling *one additional unit* after selling x units. This revenue from selling one additional unit is called the *marginal revenue*. Similarly, the marginal cost and the marginal profit at x units is the additional cost of $x+1$ units and additional profit of $x+1$ units compared to x units: $C'(x) \approx C(x + 1) - C(x)$ and $P'(x) \approx P(x + 1) - P(x)$. In economics the term "marginal" is nearly synonymous with "derivative".