Math 1131 Applications: Continuity — Fall 2019

Continuity of a function, which means $\lim_{x\to a} f(x) = f(a)$ for each a in the domain of $f(x)$, tells us that *nearby inputs in the domain always lead to nearby outputs*. This is a feature of most functions you know about: polynomials, power functions (like $\sqrt{x}$), exponential and logarithmic functions, $\sin x$, and $\cos x$ are all continuous. The notion of continuity gives a *name* to this property. Why does the property matter?

- The whole idea of approximate calculations depends on continuity. To estimate $\log_{10}\pi$, you can't plug π exactly into a computing device; calculators and computers only store numbers to a limited amount of decimals, maybe 20 digits, but π is not a 20-digit (or even 1000-digit) decimal: its decimal expansion never ends. Since $\log_{10} x$ is a continuous function, as $x \to \pi$ we have $\log_{10} x \to \log_{10}\pi$: you can *estimate* $\log_{10}\pi$ using in place of π a decimal approximation and estimate the logarithm base 10 of that approximation.

- The equations that describe how a physical system evolves have parameters. Think about what you may need to know about a falling object (its initial position and velocity) before you can predict when it reaches the ground. The final answer we are interested in (when does the object reach the ground?) depends on the parameters in the equations. In practice we can only know the parameters (or any physical measurements) approximately, so we want the solutions to the equations to have *continuous dependence on the parameters*.

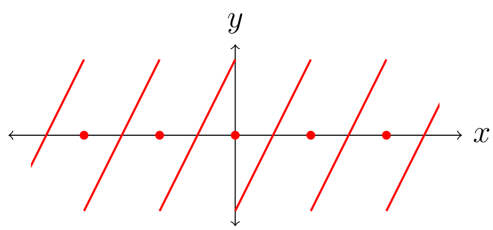
  Example. The equation $x^2 - tx + 1 = 0$ involves the parameter t. It has two solutions: $x = (t \pm \sqrt{t^2-4})/2$. Both $(t+\sqrt{t^2-4})/2$ and $(t-\sqrt{t^2-4})/2$, are continuous functions of t (if $|t| \geq 2$, so the solutions make sense in $\mathbf{R}$). This says the solutions of $x^2 - tx + 1 = 0$ have continuous dependence on t.

Continuity should be considered the very first "nice" property a function can have. There are nicer properties we can hope for (with labels like differentiability, smoothness, and analyticity; you won't see the last two in this course), but continuity is the most basic.

Remark. Another feature in a dynamical system that people care about besides continuous dependence on parameters is sensitive dependence on initial conditions, also known more colorfully as the butterfly effect. It is studied by people who work in chaos theory.

We have emphasized the importance of continuity, but discontinuities (or approximations to them) do show up in the real world. Here are examples of jump discontinuities.

1. A sawtooth wave, shown below, is a fundamental example on synthesizers and in signal processing. To hear a sawtooth wave, see the video here.



2. Electronics on the F-22 (picture below on left) shut off in 2007 when crossing the International Date Line, where longitude jumps from $-180°$ to $180°$.

3. At the edge of a shock wave (picture above on right), pressure nearly has a jump discontinuity.

4. During daylight saving time, 2 AM becomes 3 AM or *vice versa*.

5. At noon, the time label AM becomes PM. Is noon 12 AM or 12 PM? (Answer: it is 12 PM, and this is a societal convention, not something you can reason through by logic.) All morning times are AM and all afternoon times are PM, so at noon there has to be a sudden change in the label.

6. The energy levels of an electron are discrete, not continuous. Many physical quantities (energy, charge, light) that at ordinary scales appear to vary continuously at the atomic scale are discrete (physicists say “quantized”).

7. The media you use presents the illusion of continuity but is ultimately discrete (pixels on a computer screen, 24 frames per second in a movie) . An amusing example where the continuity illusion is broken is here: watch the helicopter blades!