

Passing to a limit is the characteristic feature of calculus that separates it from algebra. This entire course could be regarded as one long application of limits. We will focus here on uses of limits that are not often seen in first-semester calculus.

**Limits and irrational exponents.** In school you learn exponents in stages:

1. An exponent that is a positive integer means repeated multiplication:

$$2^m = \underbrace{2 \cdot 2 \cdots 2}_{m \text{ times}}$$

We have the rules  $2^m 2^n = 2^{m+n}$  and  $(2^m)^n = 2^{mn}$  for positive integers  $m$  and  $n$ .

2. An exponent that is 0 or a negative integer means:  $2^0 = 1$  and  $2^{-n} = 1/2^n$  for a negative integer  $-n$ . For example,  $2^{-3} = 1/2^3 = 1/8$ . *This is no longer repeated multiplication!* There is no such thing as “multiplying 2 by itself  $-3$  times.”

With these definitions, the rules  $2^m 2^n = 2^{m+n}$  and  $(2^m)^n = 2^{mn}$  are now valid when  $m$  and  $n$  are arbitrary integers (possibly 0 or negative).

3. An exponent that is rational means:  $2^{1/q} = \sqrt[q]{2}$  and  $2^{p/q} = \sqrt[q]{2^p}$  for positive integers  $q$  and any integer  $p$ . For example,  $2^{1/3} = \sqrt[3]{2}$  and  $2^{4/5} = \sqrt[5]{2^4} = \sqrt[5]{16}$ .

With this definition, the rules  $2^r 2^s = 2^{r+s}$  and  $(2^r)^s = 2^{rs}$  are valid when  $r$  and  $s$  are arbitrary rational numbers.

Rational exponents, usually in the form of finite decimals between 0 and 1, are used in [Cobb–Douglas](#) production functions in economics.

That is as far as algebra can take you. What about an irrational exponent, such as in  $2^\pi$ ? If you ask a calculator or computer for  $2^\pi$ , the answer you get is 8.8249778... *What does that actually mean?* The number  $\pi = 3.14159\dots$  is the limit of 3,  $3.1 = 31/10$ ,  $3.14 = 314/100$ ,  $3.141 = 3141/1000$ , etc. and so on. Look at this table:

$2^3$	$2^{3.1}$	$2^{3.14}$	$2^{3.141}$	$2^{3.1415}$	$2^{3.14159}$
8	8.5741...	8.8152...	8.8213...	8.8244...	8.8249...

The numbers from left to right seem to heading towards some value 8.824..., and  $2^\pi$  is defined to be that **limit**:  $2^\pi$  is the limit of  $2^{p/q}$  for rational  $p/q \rightarrow \pi$ . In the

same way,  $2^x$  for an irrational number  $x$  is the limit of the numbers  $2^{p/q}$  for rational  $p/q \rightarrow x$ . With this definition, the rules  $2^x 2^y = 2^{x+y}$  and  $(2^x)^y = 2^{xy}$  turn out to remain valid when  $x$  and  $y$  are arbitrary real numbers.

Is this useful? Yes! For example, logarithm values can be thought of as exponents ( $b^{\log_b x} = x$ ) and many logarithm values are irrational. There wouldn't be a nice graph for  $y = \log_b x$  if we didn't have a meaning for irrational exponents.

**Limits and physics.** Two ways in which limits appear in physics is (i) deriving physical laws and (ii) checking the compatibility of new physical theories with older ones under conditions where the old one fits experiments well.

### 1. Deriving physical laws.

There are many equations in physics telling us how things evolve: the heat equation, the wave equation, and so on. If you look up the *derivation* of such equations by physical reasoning, you will find a process of discretization (small intervals of length or time, say) and then a limit as the discretization tends to 0 ( $\Delta x \rightarrow 0$  or  $\Delta t \rightarrow 0$ ).

### 2. Compatibility between physical theories.

There were two revolutions in physics in the early 1900s: Newton's law of gravity was replaced by Einstein's relativity theory and classical mechanics was replaced by quantum mechanics. Two characteristic features of the new physics was the use of  $v/c$  in relativity, where  $v$  is an object's speed and  $c$  is the speed of light (no physical object can travel at that speed, so  $v/c < 1$ ), and a new physical constant  $h$  (Planck's constant) in quantum mechanics. Since Newtonian physics had been so successful at describing the universe for 200+ years before 1900, there had to be a compatibility between the old and new physics under conditions when the old physics was already experimentally well tested.<sup>1</sup> Since at ordinary scales we travel at speeds *much less* than the speed of light and Planck's constant in ordinary units is *very small* (around  $6.626 \times 10^{-34}$  Joules-sec), classical physics can be regarded as a **limiting case** of modern physics by letting  $v/c \rightarrow 0$  in relativistic formulas and  $h \rightarrow 0$  in quantum formulas.

**Example 1.** If  $P$ ,  $Q$ , and  $R$  are three particles traveling along a straight line,  $v_{PQ}$  is the velocity of  $P$  as measured by  $Q$  and  $v_{QR}$  and  $v_{PR}$  are defined similarly, then classically  $v_{PR} = v_{PQ} + v_{QR}$  (this fits our intuition – if a car travels at 40 mph on a road parallel to train tracks and a train on the tracks goes at 30 mph as measured

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<sup>1</sup>Classical physics continues to be widely used when relativistic and quantum effects are negligible. We did not need relativity or quantum mechanics to send people to the moon in 1969.

by someone in the car then the train's velocity will be measured by someone on the ground as being 70 mph) while relativistically

$$v_{PR} = \frac{v_{PQ} + v_{QR}}{1 + (v_{PQ}v_{QR}/c^2)}.$$

At ordinary speeds,  $v_{PQ}/c$  and  $v_{QR}/c$  are nearly 0, so their product  $v_{PQ}v_{QR}/c^2$  is nearly 0. Thus the relativistic formula for  $v_{PR}$  has denominator nearly 1, in which case the relativistic velocity formula turns into  $v_{PR} \approx v_{PQ} + v_{QR}$ , which is essentially the classical velocity formula.

**Example 2.** Quantum mechanics says matter has wave-like properties: wavelength, interference, *etc.*. The wavelength of matter waves is  $h/p$ , where  $h$  is Planck's constant and  $p$  is the matter's momentum (classically,  $p$  is mass times velocity). As  $h \rightarrow 0$  the wavelength  $h/p$  is negligible, so we don't see wave-like properties of bulk matter.

Strictly speaking  $h$  is a constant, so it can't literally tend to 0. Another way of describing this situation is that at ordinary scales  $h/p$  is negligible since  $h \approx 6.626 \times 10^{-34}$  J-s is so small.

Relativity and quantum mechanics have features that can't be described by classical physics, such as spacetime curvature and entangled quantum states, but those effects become negligible in the classical limit ( $v/c \rightarrow 0$  or  $h \rightarrow 0$ ).

**Limits and animation.** The Numberphile video [here](#) is an interview with Tony DeRose, the head of the Research Group at Pixar Animation, who shows how Pixar can create smooth curves as limits of polygons (by "splitting and averaging") and the analogue for surfaces, as seen below in the passage from left to right. Limits and other tools from calculus are used to generate the shapes in computer animation and to render suitably realistic motion (for clothing, hair, water, *etc.*)

