Common Parametric Surfaces

Here is a list of common surfaces and a (general) parameterization. For each example, state the parameterization that you would use and determine the bounds for the variables where appropriate. Describe the grid curves and sketch a graph of the surface with the grid curves on it.

1. **Planes**

   There are two common parameterizations for a plane $ax + by + cz = d$.

   (a) If we know three points P, Q, and R on the plane, then we can let $\vec{a}$ be the position vector of P, $\vec{b} = \overrightarrow{PQ}$, and $\vec{c} = \overrightarrow{PR}$. Then one parameterization is given by

   $$\vec{r}(u, v) = \vec{a} + \vec{b}u + \vec{c}v.$$

   (b) We can instead solve for one variable and replace the other two with u and v. Say, for example, that $c \neq 0$. Then we can solve for $z = \frac{1}{c}(d - ax - by)$. Then a second parameterization is given by

   $$\vec{r}(u, v) = \left\langle u, v, \frac{1}{c}(d - au - bv) \right\rangle.$$

   Example: $8x - y + 3z = 12$, $-1 \leqslant x \leqslant 4$, $2 \leqslant z \leqslant 5$.

2. **Elliptic Paraboloids**

   There are also two common parameterizations for an elliptic paraboloid, say $z = a(x^2 + y^2)$, $a > 0$.

   (a) Since we already have z described as a function of x and y, we can simply use the following parameterization. However, this isn't always ideal and its usefulness depends on the bounds/regions given in integrals, for example.
   $$\vec{r}(u, v) = \langle u, v, a(u^2 + v^2)\rangle$$

   (b) We could instead use cylindrical coordinates. This is often more useful, especially if any other surfaces are involved in integrals, like cylinders, cones, or spheres.
   $$\vec{r}(u, v) = \langle u\cos v, u\sin v, au^2\rangle.$$

   Example: $z = 4x^2 + 4y^2$, $z \leqslant 16$.

3. **Cylinders**

Let's assume that we have a cylinder of the form $x^2 + y^2 = a^2$. We can use cylindrical coordinates to get a quick parameterization of the form

$$\vec{r}(u, v) = \langle a \cos u, a \sin u, v \rangle, \ 0 \leqslant u \leqslant 2\pi.$$

Example: $x^2 + y^2 = 16$, $0 \leqslant z \leqslant 7$.

4. **Cones**

Much like an elliptic paraboloid, we can parameterize a cone of the form $z = a\sqrt{x^2 + y^2}$ in two ways.

(a) Since we already have z described as a function of x and y, we can just use the following parameterization. Again, this often complicates integrals, but it is still a valid parameterization nonetheless.

$$\vec{r}(u, v) = \langle u, v, a\sqrt{u^2 + v^2} \rangle$$

(b) If we instead use cylindrical coordinates, we get a parameterization that often has a more useful form, namely

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, au \rangle, \ 0 \leqslant v \leqslant 2\pi$$

Example: $z = \sqrt{3(x^2 + y^2)}$, $z \leqslant 3$.

5. **Spheres**

To parameterize a sphere, we can simply use spherical coordinates. Say that $x^2 + y^2 + z^2 = a^2$ and assume $a > 0$. Then a parameterization is given by

$$\vec{r}(u, v) = \langle a \cos u \sin v, a \sin u \sin v, a \cos v \rangle.$$

Example: $x^2 + y^2 + z^2 = 64$.

Answers

1. $\vec{r}(u,v) = \langle u, 8u + 3v - 12, v \rangle$, $-1 \leqslant u \leqslant 4$, $2 \leqslant v \leqslant 5$
2. $\vec{r}(u,v) = \langle u\cos v, u\sin v, 4u^2 \rangle$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 2\pi$
3. $\vec{r}(u,v) = \langle 4\cos u, 4\sin u, v \rangle$, $0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant 7$
4. $\vec{r}(u,v) = \langle u\cos v, u\sin v, \sqrt{3}u \rangle$, $0 \leqslant u \leqslant \sqrt{3}$, $0 \leqslant v \leqslant 2\pi$
5. $\vec{r}(u,v) = \langle 8\cos u\sin v, 8\sin u\sin v, 8\cos v \rangle$, $0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant \pi$