Double Integrals in Polar Coordinates Solutions

1. $\iint\limits_D xy\,dA = \int_0^{\frac{\pi}{2}}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta$

   $= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4}r^4\cos\theta\sin\theta\right]_1^2\,d\theta$

   $= \int_0^{\frac{\pi}{2}} \frac{15}{4}\cos\theta\sin\theta\,d\theta$

   Let $u = \sin\theta$ ($\cos\theta$ is fine too), $du = \cos\theta\,d\theta$

   $= \int_0^1 \frac{15}{4}u\,du$

   $= \left[\frac{15}{8}u^2\right]_0^1 = \frac{15}{8}$

2. (a) $\int_{-1}^0\int_0^{\sqrt{1-x^2}} 8x^3y\,dy\,dx = \int_{\pi/2}^{\pi}\int_0^1 8r^5\cos^3\theta\,\sin\theta\,dr\,d\theta$

   $= \int_{\pi/2}^{\pi} \left[\frac{4}{3}r^6\cos^3\theta\,\sin\theta\right]_0^1\,d\theta$

   $= \int_{\pi/2}^{\pi} \frac{4}{3}\cos^3\theta\,\sin\theta\,d\theta$

   Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$

   $= \int_0^{-1} -\frac{4}{3}u^3\,du$

   $= \left[-\frac{1}{3}u^4\right]_0^{-1}$

   $= -\frac{1}{3}$

   (b) $\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} e^{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_0^2 re^{r^2}\,dr\,d\theta$

   Let $u = r^2$, $du = 2r\,dr$

   $= \int_0^{\frac{\pi}{4}}\int_0^4 \frac{1}{2}e^u\,du\,d\theta$

   $= \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}e^u\right]_0^4\,d\theta$

   $= \int_0^{\frac{\pi}{4}} \frac{1}{2}e^4 - \frac{1}{2}\,d\theta$

   $= \left[(\frac{1}{2}e^4 - \frac{1}{2})\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}e^4 - \frac{\pi}{8}$

3. (a) The eighth of the circular region of radius 4 in the third quadrant with one side lying along the negative x-axis.

   (b) $\int_{\pi}^{5\pi/4}\int_0^4 \frac{3r^3\cos^2\theta}{r\sin\theta}\,dr\,d\theta$

4. Suppose we want to find the volume between the planes $z = x - y$ and $z = 0$ inside the cylinder $x^2 + y^2 = 4$.

(a) $\displaystyle\iint_D (x - y)\, dA = \int_0^{2\pi} \int_0^2 (r^2 \cos\theta - r^2 \sin\theta)\, dr\, d\theta$

$= \displaystyle\int_0^{2\pi} \left[\tfrac{1}{3} r^3 \cos\theta - \tfrac{1}{3} r^3 \sin\theta\right]_0^2 d\theta$

$= \displaystyle\int_0^{2\pi} (\tfrac{8}{3}\cos\theta - \tfrac{8}{3}\sin\theta)\, d\theta$

$= \left[\tfrac{8}{3}\sin\theta + \tfrac{8}{3}\cos\theta\right]_0^{2\pi}$

$= (0 + \tfrac{8}{3}) - (0 + \tfrac{8}{3}) = 0$

Not the volume, just the signed volume, the negative and positive parts are equal.

(b) There's a cylinder, then the plane slices at an angle through the origin, equal volumes are enclosed above and below $z = 0$. Part above $z = 0$ is below the line $y = x$, and part above $z = 0$ is above the line $y = x$. Take the volume above $z = 0$ and double it.

Volume $= 2 \displaystyle\int_{-3\pi/4}^{\pi/4} \int_0^2 (r^2 \cos\theta - r^2 \sin\theta)\, dr\, d\theta$

$= 2 \displaystyle\int_{-3\pi/4}^{\pi/4} \left[\tfrac{1}{3} r^3 \cos\theta - \tfrac{1}{3} r^3 \sin\theta\right]_0^2 d\theta$

$= 2 \displaystyle\int_{3\pi/4}^{\pi/4} (\tfrac{8}{3}\cos\theta - \tfrac{8}{3}\sin\theta)\, d\theta$

$= 2\left[\tfrac{8}{3}\sin\theta + \tfrac{8}{3}\cos\theta\right]_{-3\pi/4}^{\pi/4}$

$= 2\left[\left(\dfrac{8\sqrt{2}}{6} + \dfrac{8\sqrt{2}}{6}\right) - \left(-\dfrac{8\sqrt{2}}{6} - \dfrac{8\sqrt{2}}{6}\right)\right]$

$= \dfrac{32\sqrt{2}}{3}$