Line Integral Theorems

1. Let $\vec{F}(x,y,z)$ be a vector field and assume $\vec{F} = \vec{\nabla} f$ for some function $f(x,y,z)$. If $\int_{C_1} \vec{F} \cdot d\vec{r} = 2$ and C_1 is given by $\vec{r_1}(t) = \langle t, t^2, t^3 \rangle$, $0 \leqslant t \leqslant 1$, then what is the value of $\int_{C_2} \vec{F} \cdot d\vec{r}$, if C_2 is given by $\vec{r_2}(t) = \left\langle 1 - \frac{1}{2}t, -\frac{1}{8}(t^3 - 8), \cos(\frac{\pi t}{4}) \right\rangle$, $0 \leqslant t \leqslant 2$? Justify your answer. Hint: think about where C_1 starts and ends, and where C_2 starts and ends.

2. Let $\vec{F}(x,y) = \langle y^2, x^2 y \rangle$ and C the rectangle with vertices $(0,0)$, $(5,0)$, $(5,4)$, and $(0,4)$ and positive orientation. Consider the line integral
$$\int_C \vec{F} \cdot d\vec{r}.$$
   (a) Compute this line integral directly.
   (b) Compute this line integral using Green's Theorem.

3. Compute $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F} = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$ and C is the triangle formed by going from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

4. Suppose we have a region D in the plane with closed boundary curve C that is positively oriented, $\vec{n}$ the outward unit normal vector to the curve C, and a vector field $\vec{F} = \langle P, Q \rangle$. The following form of Green's Theorem says we can determine the outward flow of $\vec{F}$ across C (this is what integrating $\vec{F} \cdot \vec{n}$ measures) by evaluating a double integral over the enclosed region D.
$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \mathrm{div}\vec{F}\, dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)\, dA,$$
   Using this version of Green's Theorem, compute $\int_C \vec{F} \cdot \vec{n}\, ds$, if $\vec{F} = \langle xy^2, x^2 y \rangle$ and C is the curve given by $\vec{r}(t) = \langle 3\cos t, 3 \sin t \rangle$, $0 \leqslant t \leqslant 2\pi$ with positive orientation.

Answers

1. -2
2. 120
3. $-\frac{16}{3}$
4. $\frac{81\pi}{2}$