Finding Absolute Maxima and Minima

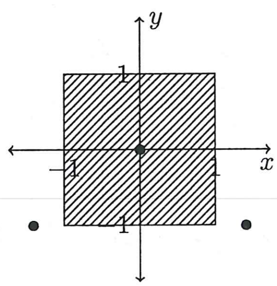
In Calculus I, we first learned how to find and classify critical points, which allow us to find the location of local maxima and minima. We also discussed the application of finding the absolute maximum and minimum values of a function $y = f(x)$ over a closed interval $[a, b]$. Recall that the absolute maximum and minimum can only occur at either critical points or at the endpoints of the interval but nowhere else, assuming that $f(x)$ is continuous on $[a, b]$. This is known as the Extreme Value Theorem, and our goal is to now extend these ideas to functions of two or more variables.

Once again, it should be clear that an absolute maximum or minimum value can occur at a critical point, but what corresponds to the endpoints? First, absolute extrema are only guaranteed to exist over a set D that is both *closed* and *bounded*. What does that mean? A set D is called closed if it includes its boundary. For example, the set defined by $x^2 + y^2 \leqslant 1$ is closed, but $x^2 + y^2 < 1$ is not (the latter set would have a dashed circle for its boundary while the first would be solid). A set D is called bounded if it doesn't extend infinitely in any direction. Mathematically, this is equivalent to being able to fit the set D inside of a disk with some finite radius.

We can now state the Extreme Value Theorem for a function of two variables. If $f(x, y)$ is continuous on a closed, bounded set D in the plane, then f will attain an absolute maximum and minimum value at either a critical point inside D or on the boundary of the set D.

Example 1: Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 + x^2y + 4$ on the set D, the rectangle given by $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$. Note: the critical points of $f(x, y)$ are $(0, 0)$, $(\sqrt{2}, -1)$, and $(-\sqrt{2}, -1)$.

The critical points $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$ are not contained in the region D, so $(0, 0)$ is the only critical point that we need to consider. $f(0, 0) = 4$, which we will use to determine the absolute minimum and maximum values.



Next, we need to determine the maximum and minimum values on each piece of the boundary curve, so there are four possible line segments to consider:

If $y = 1$, then $f(x, 1) = 2x^2 + 5$, which has a minimum on $[-1, 1]$ of $f(0, 1) = 5$ and maximum of $f(\pm 1, 1) = 7$.

If $y = -1$, then $f(x, -1) = 5$, so the value for any x in $[-1, 1]$ is 5.

If $x = 1$, then $f(1, y) = y^2 + y + 5$, which has a minimum on $[-1, 1]$ of $f(1, -1/2) = 19/4$ and maximum of $f(1, 1) = 7$.

If $x = -1$, then $f(-1, y) = y^2 + y + 5$, which has a minimum on $[-1, 1]$ of $f(-1, -1/2) = 19/4$ and maximum of $f(-1, 1) = 7$.

So, comparing the maximum and minimum values at the critical point $(0, 0)$ and around the various boundary curves, we have that the absolute maximum value of $f(x, y)$ is 7 and the absolute minimum value is 4.

In the event that the boundary curve of a set D is given by one equation, the computations can still be amazingly tedious, but we can use another method to find the absolute maximum and minimum values. This curve can be thought of as a level curve of some function $g(x, y)$, so we write the curve as $g(x, y) = k$ (some constant k). This is called a **constraint**, and solving many real-world problems goes hand-in-hand with solving constrained optimization problems.

The method that we will use is called the method of **Lagrange multipliers**, which essentially relies on the observation that a function $f(x, y)$ is maximized or minimized over a curve $g(x, y) = k$ whenever the functions' gradient vectors are parallel at a point on the curve (see the images in Section 14.8 of our textbook for a nice illustration of this). This works for functions defined with any number of variables, but we will state the system in the case of two variables. Namely, given a function $f(x, y)$ that we wish to maximize or minimize and a constraint $g(x, y) = k$, we seek solutions to the system of equations

$$\begin{aligned} \vec{\nabla} f(x,y) &= \lambda \cdot \vec{\nabla} g(x,y) \\ g(x,y) &= k \end{aligned}$$

Here, λ is a constant, called a Lagrange multiplier. Also, note that $\vec{\nabla} f = \lambda \vec{\nabla} g$ is really *two* equations: $f_x = \lambda g_x$ and $f_y = \lambda g_y$, which come from the corresponding components of the gradient vectors.

Example 2: Find the absolute maximum and minimum values of $f(x, y) = xy^2$ on $x^2 + y^2 = 3$.

We want to solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases} \Rightarrow \begin{cases} y^2 = \lambda \cdot 2x \\ 2xy = \lambda \cdot 2y \\ x^2 + y^2 = 3 \end{cases}$$

If $y^2 = \lambda \cdot 2x$, then $x = y = 0$ is a solution, but $(0,0)$ is not on the curve $x^2+y^2=3$. So, we must have $\lambda = \frac{y^2}{2x}$.

If $2xy = \lambda \cdot 2y$, then either $y = 0$ or $\lambda = x$.

$y = 0 \Rightarrow x^2 + 0 = 3 \Rightarrow x = \pm\sqrt{3} \Rightarrow (\sqrt{3}, 0), (-\sqrt{3}, 0)$

$\lambda = x \Rightarrow \underbrace{\lambda = x = \frac{y^2}{2x}}_{\text{same } \lambda \text{ for both}} \Rightarrow y^2 = 2x^2$

Substituting into $x^2+y^2=3$, we get $x^2 + 2x^2 = 3x^2 = 3 \Rightarrow x = \pm 1$

$\Rightarrow$ four points $(\pm 1, \pm\sqrt{2})$

Plugging into $f(x,y) = xy^2$:

$f(\pm\sqrt{3}, 0) = 0$, $f(1, \pm\sqrt{2}) = 2$, $f(-1, \pm\sqrt{2}) = -2$

$\boxed{\therefore \text{ Absolute max value is } 2, \text{ absolute min value is } -2}$

No critical points

Example 3: Find the maximum and minimum values of $\underbrace{f(x,y) = 3x+y}$ subject to the constraint $x^2+y^2 = 10$.

$$\begin{cases} 3 = \lambda \cdot 2x \Rightarrow \lambda = \frac{3}{2x} \\ 1 = \lambda \cdot 2y \Rightarrow \lambda = \frac{1}{2y} \\ x^2+y^2=10 \end{cases} \quad \Rightarrow \frac{3}{2x} = \frac{1}{2y} \Rightarrow 3y = x$$

Combining $3y=x$ and $x^2+y^2=10$, we have

$$x^2+y^2 = (3y)^2 + y^2 = 10y^2 = 10 \Rightarrow y = \pm 1$$

Since $3y=x$, we consider the points $(3,1), (-3,-1)$.

Evaluating $f(x,y) = 3x+y$ at these points yields

$f(3,1) = 10$ and $f(-3,-1) = -10$.

$$\boxed{\begin{array}{l} \therefore \text{ absolute max value is } 10 \\ \text{absolute min value is } -10 \end{array}}$$

Sometimes we only obtain one value from this method, meaning that there is either a maximum with no minimum or vice versa. It is then important to check analytically to determine which one has been found.

Example 4: Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$.

$$\begin{cases} 2x = \lambda \cdot y \Rightarrow \lambda = \frac{2x}{y} \\ 2y = \lambda \cdot x \Rightarrow \lambda = \frac{2y}{x} \\ xy = 1 \end{cases} \Rightarrow \frac{2x}{y} = \frac{2y}{x} \Rightarrow x^2 = y^2$$

*Note: we can ignore the point $(0,0)$ because it's not on $xy=1$

$x^2 = y^2$ is equivalent to $y = \pm x$, but since we need the product of x and y to be positive $(xy=1)$, we can ignore $y = -x$.

Geometrically, we can see this as intersections of the curve $xy = 1$ $(y = \frac{1}{x})$ with the lines $y = \pm x$.

If $xy = 1$ and $y = x$, we have $x^2 = 1$ or $(1,1)$ and $(-1,-1)$.

$f(1,1) = f(-1,-1) = 2$, but is it the absolute max or min value?

Pick another point on $xy=1$, say $(2, \frac{1}{2})$. $f(2,\frac{1}{2}) = 4 + \frac{1}{4} > 2$, so 2 is the absolute min value, and there is no absolute max.

Practice Exercises

1. Find the absolute maximum and minimum values of the function $f(x, y) = y^2 - x^2$ over the region given by $x^2 + 4y^2 \leqslant 4$. (Hint: use Lagrange multipliers to find the max and min on the boundary)
2. Find the maximum area of a rectangle with sides measuring x and y if the perimeter is 14. Is there a minimum value of the area?