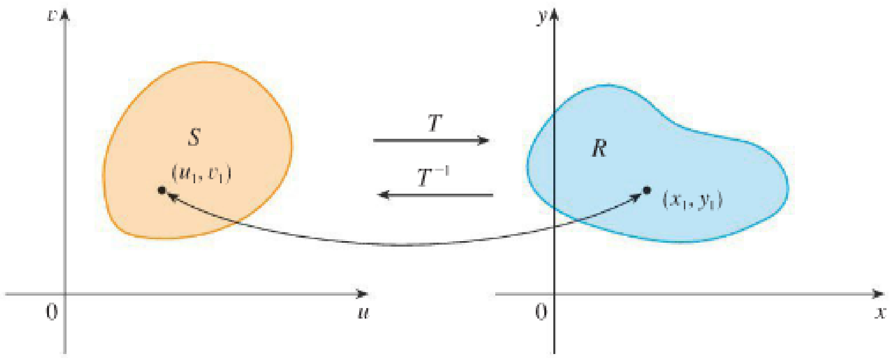
Change of Variables in Multiple Integrals

In Calculus I, a useful technique to evaluate many difficult integrals is by using a u-substitution, which is essentially a change of variable to simplify the integral. Sometimes changing variables can make a huge difference in evaluating a double integral as well, as we have seen already with polar coordinates. This is often a helpful technique for triple integrals as well.

In general, say that we have a transformation $T(u,v) = (x,y)$ that maps a region S to a region R (see picture below). All images are taken from Stewart, 8th Edition.



We define the **Jacobian** of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ as

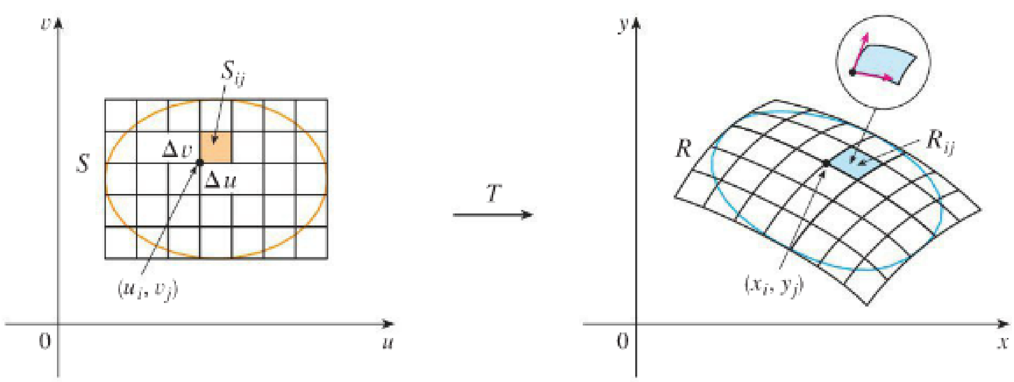
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

We can use this notation to approximate the subareas ΔA of the region R, the image of S under T:

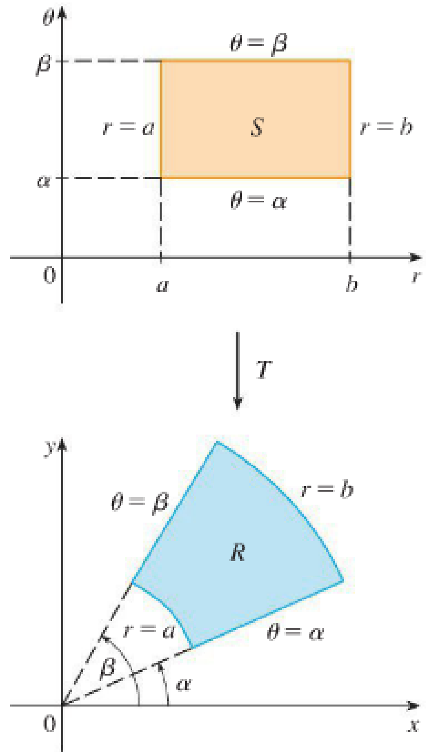
$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v.$$

Dividing the region S in the uv-plane into rectangles S_{ij} and calling their images in the xy-plane R_{ij} (see picture below), we can approximate the double integral of a function $f(x,y)$. Taking limits of the double sum, we get the following:

$$\iint_R f(x,y)\, dA = \iint_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, du\, dv$$



We have seen an example of this with polar coordinates. In that case, the transformation $T(r,\theta) = (x,y)$ is given by $x = g(r,\theta) = r\cos\theta$ and $y = h(r,\theta) = r\sin\theta$.



The Jacobian of the transformation T is given by

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, we have that

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\,dr\,d\theta = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

One way to understand the extra factor of r in the integral is to think about how the area of each region is affected if we change the bounds. If we keep the bounds on θ the same, say $\alpha \leqslant \theta \leqslant \beta$, but change the radius from $1 \leqslant r \leqslant 2$ to $101 \leqslant r \leqslant 102$, the area of the region in terms of x and y dramatically increases, even though the area of the rectangle in r and θ would be the same. In short, the bigger the radius, the bigger the area, so the area is scaled up accordingly.

The Jacobian is defined in a similar manner for a transformation with three variables, say $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$. Then we have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In particular, there are similar factors with cylindrical coordinates and spherical coordinates, two examples of three-variable transformations, which have Jacobians of r and $\rho^2\sin\phi$, respectively.

One of the most useful applications of a change of variables is simplifying otherwise complicated and/or tedious integrals. One way to do this is to look at the boundary curves of the region R and see where they are taken under the transformation T. Looking at the boundary of R allows us to determine the region S and use the Jacobian to compute the integral in a different way.

Example 1: Use the transformation given by $x = 2u + v$, $y = u + 2v$ to compute the double integral $\iint\limits_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$.

Example 2: Use the transformation given by $x = 2u$, $y = 3v$ to compute the double integral $\displaystyle\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$.

Practice Exercises

1. Find the Jacobian for each transformation.
   (a) $x = 5u - v$, $y = u + 3v$
   (b) $x = uv$, $y = u/v$
   (c) $x = e^{-r}\sin\theta$, $y = e^r\cos\theta$

2. Find the image of the set S under the given transformation.
   (a) S is the square bounded by the lines $u = 0$, $u = 3$, $v = 0$, $v = 3$; $x = 2u + 3v$, $y = u - v$
   (b) S is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$; $x = u^2$, $y = v$

3. Use the transformation given by $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$ to compute the double integral $\displaystyle\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$.