Change of Variables in Multiple Integrals

1. Find the Jacobian for each transformation.

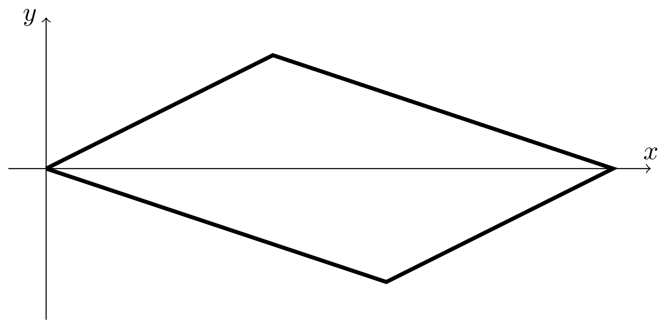
   (a) $\begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = 15 - (-1) = 16$

   (b) $\begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = (-u/v) - u/v = -2u/v$

   (c) $\begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix} = \sin^2\theta - \cos^2\theta$

2. Find the image of the set S under the given transformation.

   (a) Vertices $(0,0)$, $(3,0)$, $(3,3)$ and $(0,3)$ in xy-plane are mapped to $(0,0)$, $(6,3)$, $(15,0)$ and $(9,-3)$, respectively, in the uv-plane:

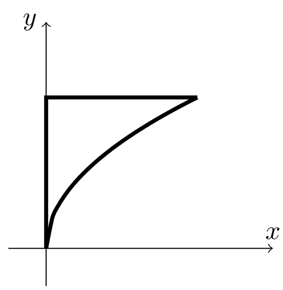
   

   (b) S is bounded by $u = 0$, $v = 1$, and $u = v$

   $u = 0 \Rightarrow x = 0$ and $y = v$ (we get $x = 0$ as a bounding curve)

   $v = 1 \Rightarrow x = u^2$ and $y = 1$ (we get $y = 1$ as a bounding curve)

   $u = v \Rightarrow x = v^2$ and $y = v \Rightarrow y = \sqrt{x}$

3. $x = \frac{1}{4}(u+v) \Rightarrow 4x = u+v \Rightarrow v = 4x - u.$

$y = \frac{1}{4}(v-3u) \Rightarrow 4y = v - 3u \Rightarrow 4y = 4x - u - 3u \Rightarrow u = x - y$

Then $v = 4x - (x-y) = 3x + y.$

The vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$ in the xy-plane are mapped to $(-4,0)$, $(4,0)$, $(4,8)$, and $(-4,8)$, respectively, in the uv-plane (giving the rectangle $-4 \leqslant u \leqslant 4$, $0 \leqslant v \leqslant 8$).

The Jacobian is $\begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = 1/16 - (-3/16) = 1/4$

$$\begin{aligned}
\iint_R (4x+8y)\,dA &= \int_{-4}^{4}\int_0^8 (u+v+2v-6u)\tfrac{1}{4}\,dv\,du \\
&= \tfrac{1}{4}\int_{-4}^{4}\left[\tfrac{3v^2}{2} - 5uv\right]_0^8 du \\
&= \tfrac{1}{4}\int_{-4}^{4} 96 - 40u\,du \\
&= \tfrac{1}{4}\left[96u - 20u^2\right]_{-4}^{4}\,du \\
&= \tfrac{1}{4}\left[(384-320) - (-384-320)\right] = 192
\end{aligned}$$