University of Connecticut
Department of Mathematics

MATH 2110Q | PRACTICE EXAM 1 | FALL 2017

NAME: SOLUTIONS

DISCUSSION SECTION: ______

**Read This First!**

- Read the questions and instructions carefully.
- The available points for each problem are given in brackets.
- You must **show your work** to obtain full credit (and to possibly receive partial credit). Correct answers with no justification will not receive credit.
- Make sure your answers are clearly indicated, and cross out any work you do not want graded.
- Do not leave any blanks! Even if you do not arrive at an answer, show as much progress towards a solution as you can, and explain your reasoning.
- Calculators are not allowed.

**Grading - For Administrative Use Only**

| Page: | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Points: | 8 | 13 | 11 | 7 | 11 | 50 |
| Score: | | | | | | |

1. Let $\vec{a} = \langle 5, -1, 2\rangle$ and $\vec{b} = \langle 3, 1, 1\rangle$. Find $\vec{a} \cdot (\vec{a} \times \vec{b})$ [3]

$\vec{a}$ is not parallel to $\vec{b}$, and $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$, so $\vec{a}\cdot(\vec{a}\times\vec{b}) = \underline{0}$.

(You can verify this through direct computation)

2. If the angle between two planes is defined as the angle between their normal vectors, find the cosine of the angle between the planes $x + y = 2$ and $x + y + \sqrt{2}z = \sqrt{6}$. [5]

$x+y=2 \Rightarrow \vec{n}_1 = \langle 1, 1, 0\rangle$

$x+y+\sqrt{2}z = \sqrt{6} \Rightarrow \vec{n}_2 = \langle 1, 1, \sqrt{2}\rangle$

$\vec{n}_1 \cdot \vec{n}_2 = 1+1+0 = 2$

$|\vec{n}_1| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$|\vec{n}_2| = \sqrt{1^2+1^2+\sqrt{2}^2} = 2$

$\therefore \cos\theta = \dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{2}{2\sqrt{2}} = \underline{\dfrac{1}{\sqrt{2}}}$

(meaning the angle between is $\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$)

3. Find and classify all critical points for the function $f(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 - xy + 2x + 5$ [8]

$$\vec{\nabla} f(x,y) = \langle f_x, f_y \rangle = \langle -x^2 - y + 2, y - x \rangle = \vec{0}$$

$$\Rightarrow f_x = -x^2 - y + 2 = 0 \qquad f_y = y - x = 0$$

$$\text{or } y = 2 - x^2 \qquad \text{or } y = x$$

Therefore, we need $y = x = 2 - x^2$ or $x^2 + x - 2 = (x+2)(x-1) = 0$.

So we have $x = 1, -2$ and critical points $(1,1)$ and $(-2,-2)$.

$f_{xx} = -2x$, $f_{xy} = -1$, and $f_{yy} = 1 \Rightarrow D(x,y) = f_{xx}f_{yy} - f_{xy}^2$
$= -2x - 1$

$D(1,1) = -3 < 0 \Rightarrow$ (1,1) is a saddle point.

$D(-2,-2) = 3 > 0$ and $f_{xx}(-2,-2) = 4 > 0 \Rightarrow (-2,-2)$ is a local min.

4. Find a vector equation for any one line that is parallel to the plane $3x - 5y + z = 10$. [5]

For a line to be parallel to the plane $3x - 5y + z = 10$, its direction vector $\vec{v}$ needs to be orthogonal to the plane's normal vector $\vec{n} = \langle 3, -5, 1 \rangle$.

There are infinitely many correct choices. For example, take $\vec{v} = \langle 5, 3, 0 \rangle$ (check that $\vec{v} \cdot \vec{n} = 0$). So, a line with this direction vector $\vec{v}$ is



$$\vec{r}(t) = \underbrace{\langle 7, \pi, -e \rangle}_{\text{any point}} + t\langle 5, 3, 0 \rangle = \langle 7 + 5t, \pi + 3t, -e \rangle.$$

5. Let $f(x,y) = x^2(y^3+1)^2 + 3x$. Find an equation of the tangent plane at the point $(1,1,7)$. [5]

$z_0$

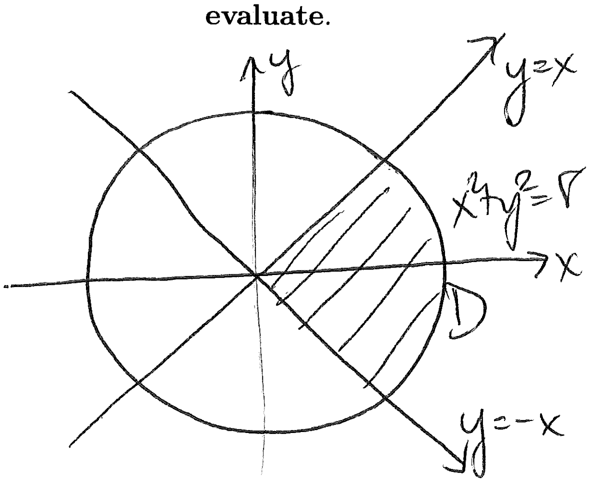
$$f_x = 2x(y^3+1)^2 + 3 \Rightarrow f_x(1,1) = 2(2)^2 + 3 = 11$$

$$f_y = 2x^2(y^3+1)\cdot 3y^2 \Rightarrow f_y(1,1) = 2(1+1)\cdot 3 = 12$$

So an equation of the tangent plane is

$$z = 7 + 11(x-1) + 12(y-1).$$

6. Let $D$ be the region in the $xy$-plane enclosed by $y = x$, $y = -x$, and $x^2 + y^2 = 8$ with $x \geq 0$. Sketch $D$ and set up a double integral in polar coordinates to compute the area of $D$. **Do not evaluate.** [6]

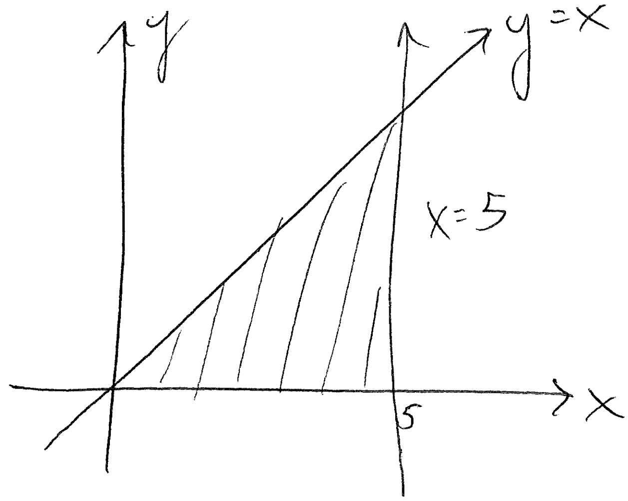



$$A(D) = \iint_D 1 \, dA$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{8}} r \, dr \, d\theta.$$

7. Consider the double integral $\int_0^5 \int_y^5 e^{x^2}\,dx\,dy$.

(a) Sketch the region being integrated over. [2]




(b) Evaluate the integral. [5]

Cannot evaluate as written ⇒ reverse the order of integration

$$\int_0^5 \int_y^5 e^{x^2}\,dx\,dy = \int_0^5 \int_0^x e^{x^2}\,dy\,dx$$

$$= \int_0^5 x e^{x^2}\,dx$$

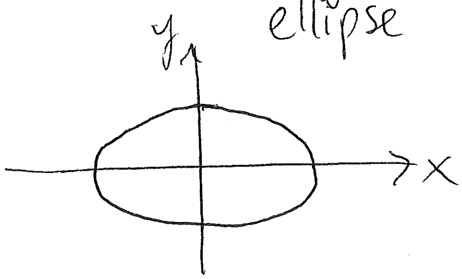
$$= \frac{1}{2} e^{x^2} \Big|_0^5$$

$$= \underline{\frac{1}{2}\left(e^{25} - 1\right)}$$

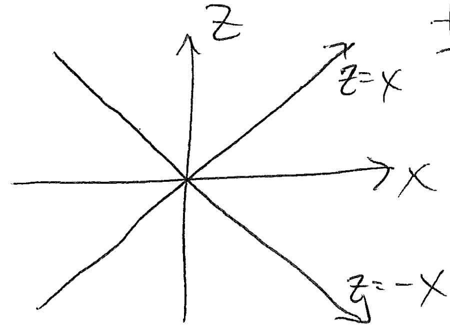
8. Give equations and sketches for two different traces of the surface $x^2 + 4y^2 - z^2 = 0$. [4]

Answers will vary, but two examples:

$z=1 \Rightarrow \underbrace{x^2+4y^2=1}_{\text{ellipse}}$




$y=0$ (xz-plane) $\Rightarrow x^2=z^2 \Rightarrow \underbrace{z=\pm x}_{\text{two lines}}$




9. Let $f(x, y) = x^2 e^{xy}$

(a) Find $D_{\vec{u}} f(2, 0)$ if $\vec{u}$ is the unit vector $\left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$. [4]

$\vec{\nabla} f(x,y) = \langle 2xe^{xy} + x^2ye^{xy}, x^3e^{xy} \rangle$

$\hookrightarrow \vec{\nabla} f(2,0) = \langle 4, 8 \rangle$

$\therefore D_{\vec{u}} f(2,0) = \vec{\nabla} f(2,0) \cdot \vec{u}$

$= \langle 4,8 \rangle \cdot \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle = \underline{\frac{28}{\sqrt{10}}}$.

(b) Find the direction in which the derivative of $f$ at $(1, 1)$ is maximized, and find the value of the maximum derivative. [3]

$\vec{\nabla} f(1,1) = \langle 3e, e \rangle$, so the direction of maximum increase is $\langle 3e, e \rangle$ (or $\langle 3, 1 \rangle$ or any other positive multiple of $\langle 3e, e \rangle$).

The maximum rate of change is

$|\vec{\nabla} f(1,1)| = \sqrt{9e^2+e^2} = \underline{\sqrt{10}\, e}$.