Review for Final Exam (Solutions)

1) **False.** $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$, assuming both are nonzero.

2) $\vec{r}(t) = \langle 1,0,2\rangle + t\underbrace{\langle 3,-1,2\rangle}_{\text{direction vector}}$

If $\vec{r}(t)$ is parallel to the plane, its direction vector is orthogonal to the normal vector of the plane, which is $\langle 1,1,-1\rangle$.

$\langle 3,-1,2\rangle\cdot\langle 1,1,-1\rangle = 3-1-2=0 \Rightarrow$ **True.**

3) $f(x,y) = \ln(x^2y^3) = 2\ln x + 3\ln y$

$f_x = \frac{2}{x}$ and $f_y = \frac{3}{y}$, so $f_{xy} = f_{yx} = 0 \Rightarrow$ **True.**

4) **False.** If $D(a,b)=0$, the point (a,b) may not be any of those types of critical points.

For example, the point $(0,0)$ for $f(x,y)=x^3$.

5) **True.** The outer bounds are over a rectangle and can be interchanged.

6) **False.** $z=r^2=x^2+y^2$ is a paraboloid.

$z=r=\sqrt{x^2+y^2}$ would be a cone.

7) <u>False</u>. The Jacobian is incorrect and should be $\rho^2 \sin\phi$, not $\rho \sin^2\phi$.

8) <u>False</u>. $\int_C \vec{F} \cdot d\vec{r} = 0$ for any <u>closed</u> curve C, but not any arbitrary curve.

9) <u>True</u>. The orientation of the curve C does not affect the value of a scalar line integral. Similarly, the orientation of a surface S does not affect the value of a scalar surface integral.

10) <u>True</u>. If $\vec{F}$ is conservative, curl $\vec{F} = \vec{0}$.

11) $z = x+y \Rightarrow x+y-z=0 \Rightarrow$ normal vector is $\langle 1,1,-1 \rangle$

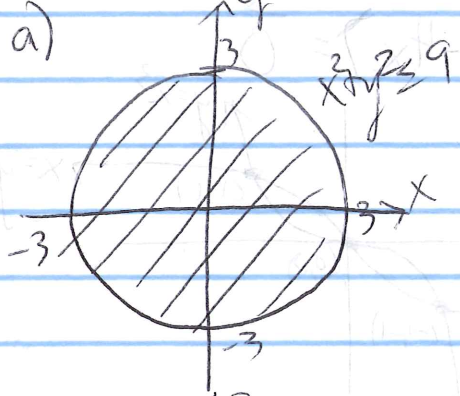
For a line to be orthogonal to the plane, its direction should be parallel to $\langle 1,1,-1 \rangle$.

So, take $\vec{r}(t) = \langle 3,-2,8 \rangle + t\langle 1,1,-1 \rangle$
$= \underline{\langle 3+t, -2+t, 8-t \rangle}$.

12) $x^2+y^2=1 \Rightarrow x^2+y^2+z^2 = z^2+1 = 4 \Rightarrow z^2 = 3 \Rightarrow z = \pm\sqrt{3}$.

So, the intersection is two circles, namely
$\underline{x^2+y^2=1,\ z=\sqrt{3}}$ and $\underline{x^2+y^2=1,\ z=-\sqrt{3}}$

13) a)



b) z-traces $\Rightarrow x^2+y^2=9-k^2$, $z=k$
circles of radius $\sqrt{9-k^2}$
x-traces $\Rightarrow$ semi-circles (vertical)
$z=\sqrt{9-k^2-y^2}$
y-traces $\Rightarrow$ same, other direction
$z=\sqrt{9-k^2-x^2}$

c)

It is the upper half of a sphere of radius 3.

14) Direction of fastest increase = direction of gradient vector

$$\vec{\nabla} C(x,y) = \langle .2x e^{.1(x^2+2y^2)}, .4y e^{.1(x^2+y^2)} \rangle$$

$$\Rightarrow \vec{\nabla} C(1,1) = \langle .2e^{.3}, .4e^{.3} \rangle$$

(or, more simply the $\langle 1,2 \rangle$ direction)

15) $z_0 = -\sin(-1+1) = 0$ ✓

$$\frac{\partial z}{\partial x} = \sin(x+y) + x\cos(x+y) \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{\substack{x=-1\\y=1}} = \sin 0 - \cos 0 = -1$$

$$\frac{\partial z}{\partial y} = x\cos(x+y) \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{\substack{x=-1\\y=1}} = -\cos 0 = -1$$

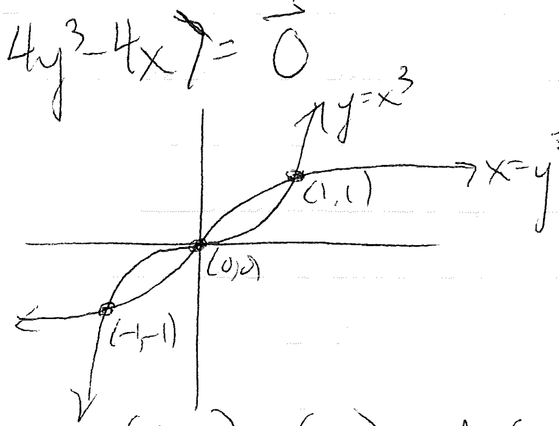
The tangent plane is $z = z_0 + \frac{\partial z}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial z}{\partial y}(x_0,y_0)(y-y_0)$

$$\Rightarrow z = 0 + (-1)(x+1) + (-1)(y-1)$$

$$\Rightarrow \underline{z = -x-y}.$$

16) $\vec{\nabla} f(x,y) = \langle 4x^3 - 4y, 4y^3 - 4x \rangle = \vec{0}$

$\Rightarrow \begin{cases} y = x^3 \\ x = y^3 \end{cases}$



The critical points are $(-1,-1)$, $(0,0)$, and $(1,1)$.

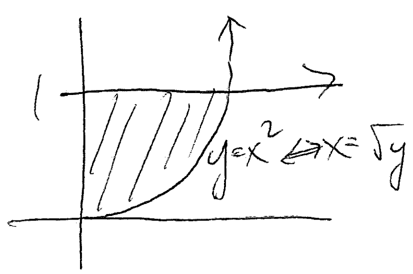
$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = (12x^2)(12y^2) - (-4)^2$
$= 144x^2y^2 - 16$

$D(-1,-1) = 144 - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0 \Rightarrow$ $\underline{(-1,-1) \text{ local min}}$

$D(0,0) = -16 < 0 \Rightarrow$ $\underline{(0,0) \text{ saddle}}$

$D(1,1) = 144 - 16 > 0$ and $f_{xx}(1,1) = 12 > 0 \Rightarrow$ $\underline{(1,1) \text{ local min}}$

17)



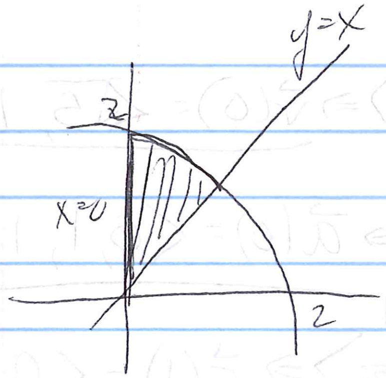
$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$

$= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$

$= \frac{1}{4} \int_0^1 \left(x^4 \sin(y^3) \Big|_0^{\sqrt{y}} \right) dy = \frac{1}{4} \int_0^1 y^2 \sin(y^3)\, dy$

$= -\frac{1}{12} \cos(y^3) \Big|_0^1 = \underline{-\frac{1}{12}(\cos(1) - 1)}$.

18)

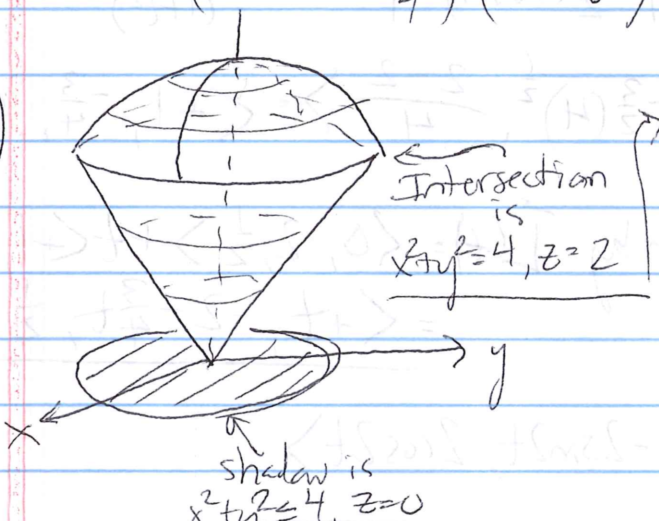


$$\iint_R 3xy\,dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 3r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= 3\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta\right)\left(\int_0^2 r^3\,dr\right)$$

$$= 3\left(\frac{1}{2}\sin^2\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right)\left(\frac{1}{4}r^4\Big|_0^2\right) = \frac{3}{8}\left(1-\frac{1}{2}\right)(\cancel{8}\ 16) = \underline{3}.$$

19)



$z=\sqrt{x^2+y^2} \Rightarrow x^2+y^2+z^2=8$
$\Rightarrow x^2+y^2+(x^2+y^2)=8$
$\Rightarrow x^2+y^2=4$
and $z=\sqrt{x^2+y^2}=\sqrt{4}=2$

Cartesian:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} xz\,dz\,dy\,dx$$

Cylindrical:

$z=\sqrt{8-x^2-y^2}=\sqrt{8-r^2}$, $z=\sqrt{x^2+y^2}=r$

$$\int_0^{2\pi}\int_0^2\int_r^{\sqrt{8-r^2}} (r\cos\theta)(z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{8-r^2}} r^2\cos(\theta)z\,dz\,dr\,d\theta$$

Spherical:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} \underbrace{\rho^4\cos\theta\sin^2\phi\cos\phi}_{(\rho\cos\theta\sin\phi)(\rho\cos\phi)(\rho^2\sin\phi)}\,d\rho\,d\phi\,d\theta$$

20) $\vec{v}(t) = \vec{s}'(t) = \langle \sqrt{5}, e^{t}, -e^{-t} \rangle \Rightarrow \underline{\vec{v}(0) = \langle \sqrt{5}, 1, -1 \rangle}$

$\vec{a}(t) = \vec{v}'(t) = \langle 0, e^{t}, e^{-t} \rangle \Rightarrow \underline{\vec{a}(0) = \langle 0, 1, 1 \rangle}$

21) $\vec{r}(t) = \langle t\ln t, -\sqrt{3t+1}, \frac{e^{t-1}}{1+t^2} \rangle \Rightarrow \vec{r}(\underline{1}) = \langle 0, -2, \frac{1}{2} \rangle$

$\vec{r}'(t) = \langle \ln t + 1, -\frac{3}{2}(3t+1)^{-\frac{1}{2}}, \frac{(1+t^2)e^{t-1} - e^{t-1}\cdot 2t}{(1+t^2)^2} \rangle$

$\vec{r}'(1) = \langle 0+1, -\frac{3}{2}(4)^{-\frac{1}{2}}, \frac{2-2}{4} \rangle = \langle +1, -\frac{3}{4}, 0 \rangle$

$\therefore$ tangent line is given by $\vec{L}(t) = \langle 0, -2, \frac{1}{2} \rangle + t\langle +1, -\frac{3}{4}, 0 \rangle$
$= \underline{\langle +t, -2-\frac{3}{4}t, \frac{1}{2} \rangle}.$

22) $\vec{r}'(t) = \langle 3\sqrt{t}, -2\sin 2t, 2\cos 2t \rangle$

$\Rightarrow |\vec{r}'(t)| = \sqrt{9t + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{9t+4}$

$L = \int_0^{\frac{1}{3}} \sqrt{9t+4}\, dt = \frac{2}{27}(9t+4)^{\frac{3}{2}} \Big|_0^{\frac{1}{3}} = \underline{\frac{2}{27}(7^{\frac{3}{2}} - 8)}.$

23) $\int_C \vec{F}\cdot dr \rightarrow$ <u>basic line integral of vector field</u>

$\int_C \text{curl}\vec{F}\, ds \rightarrow$ <u>makes no sense</u>, curl$\vec{F}$ is not a scalar

$\int_C \vec{\nabla}\cdot\vec{F}\, ds = \int_C \text{div}\vec{F}\, ds \rightarrow$ <u>makes sense</u>

$\int_C \text{div}\vec{F}\cdot d\vec{r} \rightarrow$ <u>makes no sense</u> since div$\vec{F}$ is a scalar

$\int_C \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})\, ds = \int_C \underbrace{\text{div}(\text{curl}(\vec{F}))}_{=0} ds = \underline{0}$, any vector field $\vec{F}$

24) $\vec{F} = \langle x, y, z \rangle$ is conservative and C is closed, so

$$\oint_C \vec{F} \cdot d\vec{r} = \underline{0}.$$

$$\int_C \vec{\nabla} \cdot \vec{F}\, ds = \int_C \operatorname{div}\vec{F}\, ds = \int_C 3\, ds = 3(\text{length of } C) = 3 \cdot 2\pi \cdot 1 = \underline{6\pi}.$$

$$\int_C \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F})\, ds = \underline{0} \quad \text{(see 23)}$$

25) a) $$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 & z^3 + 2x^2y & 3yz^2 \end{vmatrix} = \langle 3z^2 - 3z^2, -(0-0), 4xy - 4xy \rangle = \underline{\vec{0}}.$$

b) $\operatorname{div}\vec{F} = \underline{2y^2 + 2x^2 + 6yz}.$

c) $\underline{\text{Yes}}$, since $\operatorname{curl}\vec{F} = \vec{0}$.

d) Yes, $\vec{F}$ conservative $\Rightarrow \vec{F} = \vec{\nabla} f$ for some $f(x,y,z)$.
In particular, we can use $\underline{f(x,y,z) = yz^3 + x^2y^2}$.

e) $\vec{r}(0) = \langle 0,0,0 \rangle$, $\vec{r}(2) = \langle 4, -2, 0 \rangle$, and $\vec{F} = \vec{\nabla} f$, so by the Fundamental Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(4,-2,0) - f(0,0,0)$$

$$= (0 + 16 \cdot 4) - (0+0) = \underline{64}$$

26) $C: \vec{r}(t) = \langle 0,0,0 \rangle + t\langle 1,2,3 \rangle = \langle t, 2t, 3t \rangle, \ 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1,2,3 \rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{1+4+9} = \sqrt{14}$.

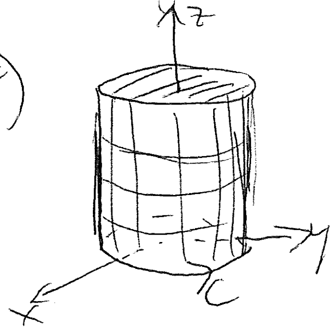
$$\int_C ze^y \, ds = \int_0^1 3te^{2t} \cdot \sqrt{14}\, dt = \dots = \underline{\frac{3\sqrt{14}}{4}(1+e^2)}.$$

27) If $\operatorname{curl}\vec{F} = \vec{0}$, $\vec{F} = \vec{\nabla} f = \langle f_x, f_y, f_z \rangle$, some $f(x,y,z)$.

$\operatorname{div}\vec{F} = \underline{f_{xx} + f_{yy} + f_{zz} = 0}$.

Several things work, but an easy example is
$f(x,y,z) = x^2 + y^2 - 2z^2 \Rightarrow \vec{F} = \underline{\langle 2x, 2y, -4z \rangle}$.

28)



Use Stokes' Theorem with boundary curve $x^2+y^2=1$, $z=0$.

$C: \vec{r}(t) = \langle \cos t, \sin t, 0 \rangle, \ 0 \le t \le 2\pi$
$\hookrightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$

$$\iint_{S} \operatorname{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 0, \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} \sin t \cos t \, dt = \frac{1}{2}\sin^2 t \Big|_0^{2\pi} = 0.$$

$\left(\begin{array}{c} \text{Also, } \operatorname{curl}\vec{F} = \vec{0}, \\ \text{so } \iint_{S} \operatorname{curl}\vec{F} \cdot d\vec{S} \\ = 0 \end{array} \right)$

29) S is closed, so we can use Divergence Theorem.

$$\oiint_{S} \operatorname{curl}\vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}(\operatorname{curl}\vec{F})}_{=0} \, dV = \underline{0}.$$

30) S' is top half of a sphere $\Rightarrow$ <u>not closed</u>.

- $\iint_{S'} \vec{F} \cdot d\vec{S}$:

a) Use Divergence Theorem $\Rightarrow$ add bottom with downward orientation

$$\oiint_{S'} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3\, dV = 3V(E) = 3 \cdot \tfrac{4}{3}\pi \cdot 1^3 = \cancel{4\pi}\ \underline{2\pi}.$$

b) Evaluate over bottom with downward orientation

Bottom: $x^2+y^2 \le 1,\ z=0 \Rightarrow \vec{r}(u,v) = \langle u\cos v, u\sin v, 0\rangle$
$0 \le u \le 1,\ 0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$
$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$ $\Rightarrow \vec{r}_u \times \vec{r}_v = \underbrace{\langle 0,0,u\rangle}_{\text{points up, wrong orientation}}$

$$-\iint_{\text{bottom}} \vec{F} \cdot d\vec{S} = -\int_0^{2\pi}\int_0^1 \langle \overset{x+z}{\overset{\parallel}{u\cos v}}, \overset{y+z}{\overset{\parallel}{u\sin v}}, \overset{z}{\overset{\parallel}{0}}\rangle \cdot \langle 0,0,u\rangle\, du\, dv$$

$= 0$

c) $\iint_{S'} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{\text{bottom}} \vec{F} \cdot d\vec{S} = \overset{2\pi}{\cancel{4\pi}} - 0 = \cancel{4\pi}\ \underline{2\pi}.$

- $\iint_{S'} \text{curl}\,\vec{F} \cdot d\vec{S}$:

Use Stokes' Theorem with counterclockwise circle $x^2+y^2=1$

$C: \vec{r}(t) = \langle \cos t, \sin t, 0 \rangle, \quad 0 \le t \le 2\pi$

$\hookrightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$

$$\iint_{S'} \text{curl}\,\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{\langle \cos t, \sin t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle}_{=0} \, dt$$

$$= \underline{0}.$$