Review for Final Exam

Questions 1-10 are true/false. Explain your reasoning in either case, and make corrections to make the statement true if it is false.

1. For any two vectors $\vec{u}$ and $\vec{v}$, then $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$.
2. The line $\vec{r}(t) = \langle 1+3t, -t, 2+2t \rangle$ is parallel to the plane $x + y - z = 4$.
3. For $f(x,y) = \ln(x^2y^3)$, $f_{xy} = f_{yx} = 0$.
4. If $f(x,y)$ is differentiable and $\vec{\nabla} f(a,b) = \vec{0}$, then (a,b) is a local maximum, local minimum, or saddle.
5. $$\int_0^1 \int_{-1}^1 \int_{xy}^{x+y} f(x,y,z)\ dz\ dy\ dx = \int_{-1}^1 \int_0^1 \int_{xy}^{x+y} f(x,y,z)\ dz\ dx\ dy$$
6. The surface $z = r^2$ is a cone in cylindrical coordinates.
7. The iterated integral
$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho sin^2\phi\ d\rho\ d\theta\ d\phi$$
represents the volume of the portion of the sphere of radius 2 in the first octant.
8. If $\vec{F} = \vec{\nabla} f$ for some function f, then for any curve C we have
$$\int_C \vec{F} \cdot d\vec{r} = 0.$$
9. If $-C$ denotes the curve C traced with opposite orientation, then
$$\int_{-C} f(x,y)\ ds = \int_C f(x,y)\ ds.$$
10. If $\vec{F}$ is a conservative vector field and S is a surface with closed boundary curve C, then
$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = 0.$$
11. Find an equation of the line through the point $(3,-2,8)$ that is orthogonal to the plane $z = x + y$.
12. Determine and describe the intersection of the surfaces $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$.
13. Let $f(x,y) = \sqrt{9 - x^2 - y^2}$.
    (a) Sketch the domain of f.
    (b) Describe the traces of f.
    (c) Sketch the graph of f. What type of surface is this?
14. Marine biologists have determined that when a shark detects blood in the water, it will swim in the direction in which the concentration of blood increases the fastest. If the concentration at any point is approximated by $C(x,y) = e^{.1(x^2+2y^2)}$, in which direction will the shark move at the point $(1,1)$?
15. Find an equation of the tangent plane to the surface $z = x\sin(x+y)$ at the point $(-1,1,0)$.
16. Find and classify all critical points for the function $f(x,y) = x^4 + y^4 - 4xy + 1$.

17. Evaluate the iterated integral

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx.$$

18. Let R be the region in the first quadrant enclosed by $x^2 + y^2 = 4$, $x = 0$, and $y = x$. Evaluate the double integral

$$\iint_R 3xy\, dA.$$

19. Let E be the region contained between $x^2 + y^2 + z^2 = 8$ and $z = \sqrt{x^2 + y^2}$. Set up but do NOT evaluate the following integral in Cartesian, cylindrical, and spherical coordinates:

$$\iiint_E xz\, dV.$$

20. If a particle's position is given by the curve $\vec{s}(t) = \langle \sqrt{5}t, e^t, e^{-t} \rangle$, find the velocity and acceleration at $t = 0$.

21. Find an equation of the tangent line at the point $(0, -2, 1/2)$ for the curve $\vec{r}(t) = \left\langle t \ln t, -\sqrt{3t+1}, \dfrac{e^{t-1}}{1+t^2} \right\rangle$.

22. Find the length of the curve C given by $\vec{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$, $0 \leqslant t \leqslant 1/3$.

23. Which of the following integrals makes sense? Explain why or why not.

$$\int_C \vec{F} \cdot d\vec{r} \qquad \int_C \text{curl}\vec{F}\, ds \qquad \int_C \vec{\nabla} \cdot \vec{F}\, ds \qquad \int_C \text{div}\vec{F} \cdot d\vec{r} \qquad \int_C \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F})\, ds$$

24. Assuming $\vec{F} = \langle x, y, z \rangle$ and C is the unit circle $x^2 + y^2 = 1,\ z = 0$, traversed counterclockwise, compute any integral that made sense in the previous question.

25. Let $\vec{F}(x, y, z) = \langle 2xy^2, z^3 + 2x^2y, 3yz^2 \rangle$.
    (a) Find curl$\vec{F}$.
    (b) Find div$\vec{F}$.
    (c) Is $\vec{F}$ conservative? Explain your answer.
    (d) Can you find a function $f(x, y, z)$ so that $\vec{\nabla} f = \vec{F}$? Explain your answer.
    (e) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ if C is given by

$$\vec{r}(t) = \left\langle t^2 e^{\cos(\pi t/4)}, te^{t-2} \cos\left(\frac{\pi}{2}t\right), e^{t-2} \sin\left(\frac{\pi}{2}t\right) \right\rangle,\ 0 \leqslant t \leqslant 2.$$

26. If C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$, evaluate

$$\int_C ze^y\, ds.$$

27. Determine an example of a non-constant vector field $\vec{F}$ that has both curl$\vec{F} = \vec{0}$ AND div$\vec{F} = 0$.

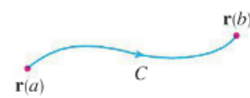
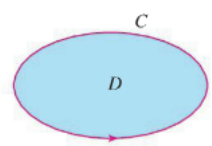
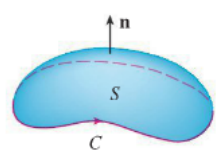
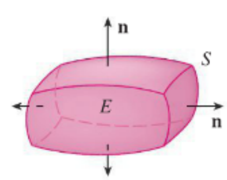
28. Compute $\iint_S \text{curl}\vec{F} \cdot d\vec{S}$ if $\vec{F} = \langle z, y, x \rangle$ and S is given by $x^2 + y^2 = 1,\ 0 \leqslant z \leqslant 4$ and $x^2 + y^2 \leqslant 1,\ z = 4$.

29. Compute $\displaystyle\iint_S \text{curl}\vec{F} \cdot d\vec{S}$ if $\vec{F} = \langle x^2 - y^2z, xy^3 + z, z^3 - (x^2 + y^2)\rangle$ if S is the surface given by $z^2 = x^2 + y^2,\ 0 \leqslant z \leqslant 3$ and $x^2 + y^2 \leqslant 9,\ z = 3$.

    Hint: Think about what the surface looks like. Which theorem(s) could be applied?

30. If $\vec{F}(x, y, z) = \langle x + z, y + z, z\rangle$ and S is the surface given by $x^2 + y^2 + z^2 = 1$ with $z \geqslant 0$ with upward orientation, compute BOTH of the surface integrals

$$\iint_S \vec{F} \cdot d\vec{S} \qquad \text{and} \qquad \iint_S \text{curl}\vec{F} \cdot d\vec{S}.$$

Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_E \text{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

Math 2110Q: Helpful Formulas

1. **The Second Derivative Test**

   Let (a, b) be a critical point of a function $f(x, y)$ with $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$.

   1. If $D(a, b) > 0$, then (a, b) is either a local maximum or minimum
      (a) $f_{xx}(a, b) < 0 \Rightarrow (a, b)$ is a local maximum
      (b) $f_{xx}(a, b) > 0 \Rightarrow (a, b)$ is a local minimum
   2. $D(a, b) < 0 \Rightarrow (a, b)$ is a saddle
   3. $D(a, b) = 0 \Rightarrow$ the test is inconclusive

2. **Summary of Line Integrals and Surface Integrals**

LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{r}(t),\ a \leqslant t \leqslant b$	$S : \vec{r}(u, v),\ (u, v) \in D$
$ds = \lvert\vec{r}\,'(t)\rvert\ dt$ =arc length differential	$dS = \lvert\vec{r}_u \times \vec{r}_v\rvert\ dA$= surface area differential
$\int_C f(x, y, z)\ ds = \int_a^b f(\vec{r}(t))\lvert\vec{r}\,'(t)\rvert\ dt$ (independent of orientation of C)	$\iint_S f(x, y, z)\ dS = \iint_D f(\vec{r}(u, v))\lvert\vec{r}_u \times \vec{r}_v\rvert\ dA$ (independent of orientation of S)
$d\vec{r} = \vec{r}\,'(t)\ dt$	$d\vec{S} = (\vec{r}_u \times \vec{r}_v)\ dA$
$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ (depends on orientation of C)	$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA$ (depends on orientation of S)
Theorems that *may* apply: Fundamental Theorem for Line Integrals Green's Theorem	Theorems that *may* apply: Stokes' Theorem Divergence Theorem