§ 14.2 Limits

Say that we have a function $y = f(x)$. If we pick a value $x = a$, we can take the limit as x approaches a from either the left or the right, and we say that the limit exists if these two values are equal. Also recall that such a function is said to be continuous at $x = a$ if the limit exists and is equal to the function's value at a.

For a function $f(x, y)$ we write the limit as the *point* (x, y) approaches the *point* (a, b), that is

$$\lim_{(x,y)\to(a,b)} f(x, y).$$

The difference is that there are an infinite number of directions and ways to approach the point (a, b). We say that the limit exists and is equal to a value L if it is the same along any path C approaching (a, b). In particular, the limit does not exist if we can find two different curves approaching (a, b) that approach different values.

Example 1: For $f(x, y) = \dfrac{x + 2y}{x + y}$, does the limit exist at the point $(0, 0)$?

If we approach the point $(0, 0)$ along the x-axis, meaning $y = 0$, we get

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,0)\to(0,0)} \frac{x + 0}{x + 0} = 1.$$

However, if we approach along the y-axis, namely $x = 0$, we get

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(0,y)\to(0,0)} \frac{0 + 2y}{0 + y} = 2.$$

Since the limits are not equal in two different directions, then the limit does not exist at the point $(0, 0)$.

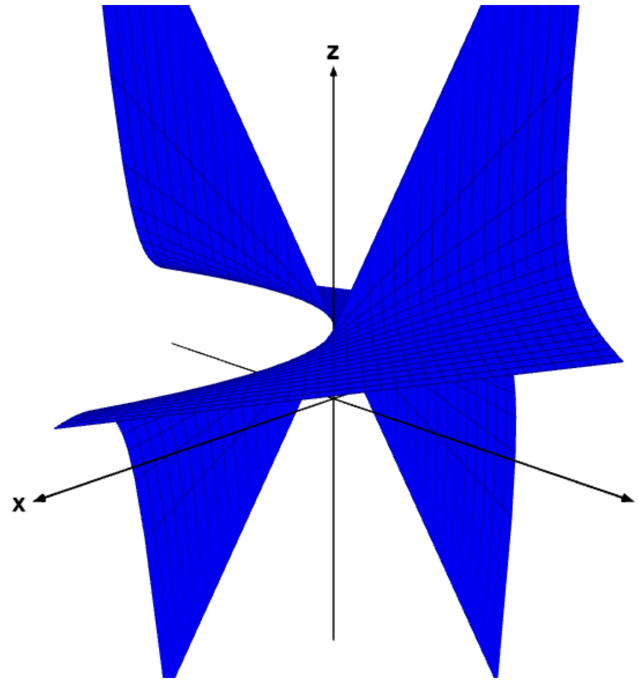

Figure 1: The function $f(x, y)$ near the point $(0, 0)$. Image courtesy of desmos.com.

But what if the limits are the same in the x- and y-directions? Does that mean that the limit exists?

Example 2: Does the limit exist at $(0,0)$ for the function $f(x,y) = \dfrac{2xy}{x^2+2y^2}$?

Start by finding the limits along the x-axis and y-axis:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{0}{x^2+0} = 0.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \frac{0}{0+2y^2} = 0.$$

However, we can also approach the point $(0,0)$ along the line $y = x$, what do we get?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} \frac{2x^2}{x^2+2x^2} = \lim_{x\to 0} \frac{2x^2}{3x^2} = \frac{2}{3} \neq 0.$$

In fact, if we approach along any line through the origin $y = mx$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{2mx^2}{x^2+2m^2x^2} = \lim_{x\to 0} \frac{2m}{1+2m^2} = \frac{2m}{1+2m^2} \neq 0 \ (m \neq 0).$$

Therefore, the limit does not exist at the point $(0,0)$.

Example 3: Compute the limit as $(x,y) \to (0,0)$ for $f(x,y) = \dfrac{x^4-4y^4}{x^2+2y^2}$ if it exists.

$$\frac{x^4-4y^2}{x^2+2y^2} = \frac{(x^2-2y^2)(x^2+2y^2)}{x^2+2y^2} = x^2-2y^2,$$

so we observe that there is a hole in the graph at $(0,0)$, but since the limit is unaffected by a hole,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x^2-2y^2 = 0.$$

Therefore, even though $(0,0)$ is not in the domain of the function $f(x,y)$, the limit exists at $(0,0)$ and is equal to 0.

Sometimes approaching a point along linear paths isn't enough to show that a limit does not exist. Take this next function, for example.

Example 4: Find the limit as $(x,y) \to (0,0)$ for $f(x,y) = \dfrac{xy^4}{x^2+y^8}$ if it exists.

If we approach $(0,0)$ in a linear direction, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{m^4x^5}{x^2+m^8x^8} = \lim_{x\to 0} \frac{m^4x^3}{1+m^8x^6} = 0.$$

However, if we approach $(0,0)$ along the path $x = y^4$, what happens?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(y^4,y)\to(0,0)} \frac{y^8}{2y^8} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2} \neq 0,$$

so the limit at $(0,0)$ does not exist.