§ 12.3 Dot Products, § 12.4 Cross Products

How do we multiply vectors?

How to multiply vectors is not at all obvious, and in fact, there are two different ways to make sense of vector multiplication, each with a different interpretation. First, we will discuss the dot product.

The Dot Product

Given two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, we define their dot product as

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Note that the dot product is a ____________________, since it has only magnitude and no direction.

Example 1: If $\vec{a} = \langle 1, -4, 3 \rangle$, $\vec{b} = \langle 2, 2, 0 \rangle$, and $\vec{c} = \langle -4, 1, -5 \rangle$, compute the following dot products:

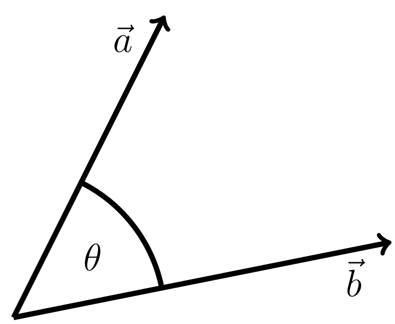
(a) $\vec{a} \cdot \vec{b}$

(b) $\vec{b} \cdot \vec{c}$

(c) $\vec{c} \cdot \vec{b}$

(d) $\vec{c} \cdot \vec{a}$

An interesting and often useful application of the dot product is finding the angle between two vectors.



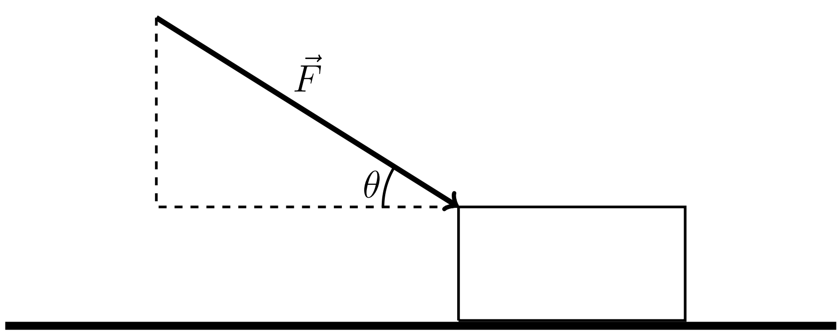
How are the two vectors related if $\theta = 0$? What about if $\theta = \pi$? Draw a picture for each situation.

What about if $\theta = 5\pi/4$? We could have just as easily described the angle between these vectors using $\theta = 3\pi/4$ instead, so in general we take $0 \leqslant \theta \leqslant \pi$.

Also, assuming $0 \leqslant \theta \leqslant \pi$, we can relate the angle between the vectors and their dot product in the following way:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta.$$

What is a familiar application of the dot product with a physical interpretation? Say that a block of a given mass is being pushed across the floor by a force of magnitude 10 N from an angle of 30° above the horizon. How much work is being done on the block if it is moved 15 m?



We can find the magnitude of the force in the direction of the block by taking $|\vec{F}|\cos\theta$, which means that the work done on the block is $W = \left(|\vec{F}|\cos\theta\right) d = \vec{F}\cdot\vec{d}$.

Example 2: Find the angle between the vectors $\langle 1, 0, 1\rangle$ and $\langle -1, 0, 0\rangle$.

The last common use of the dot product is to determine if two vectors are ______________________, meaning that the angle between them is $\theta = \pi/2$.

What is the value of the dot product when two vectors are orthogonal?

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\pi/2) = |\vec{a}||\vec{b}|\cdot 0 = 0$$

Example 3: Are the vectors $\langle 2, 1, 2\rangle$ and $\langle -1, 2, 3\rangle$ orthogonal? Explain.

The Cross Product

Before we can define the cross product, we need to briefly discuss the concept of determinant of a matrix. The determinant of a 2×2 matrix can be found as follows:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If instead our matrix is 3×3, then the determinant is much more complicated.

$$\det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

With that in mind, given two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then we define their cross product as

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \hat{\mathbf{i}} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \langle v_2 w_3 - v_3 w_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - v_2 w_1 \rangle$$

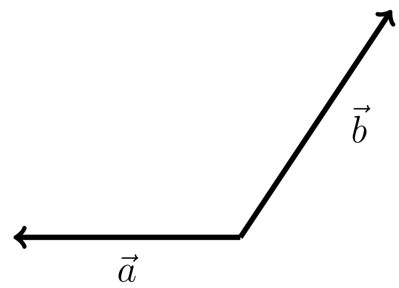
Note that the cross product is a ______________________, since it has both magnitude and direction.

Example 4: Given the vectors $\vec{v} = \langle 1, 3, 2 \rangle$ and $\vec{w} = -2, 1, 2$, compute $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$. How do these vectors compare?

In general, we have that $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$, meaning that these vectors point in opposite directions.

What is the significance of the cross product? The cross product of two vectors $\vec{v}$ and $\vec{w}$ produces a vector that is orthogonal to **both** $\vec{v}$ AND $\vec{w}$. You can determine the direction that the cross product will point using the Right-hand Rule.

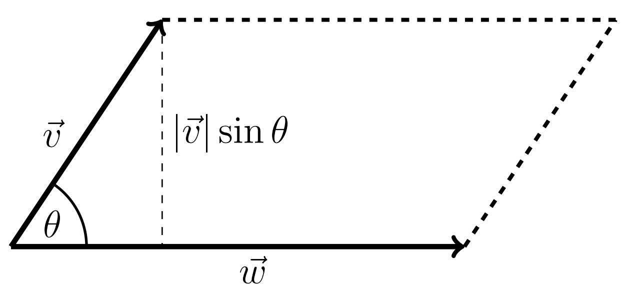
Example 5: Say that the following vectors are in the xy-plane (the paper). In what direction will the cross product $\vec{a} \times \vec{b}$ point and why?



Much like the dot product, the cross product can be related to the angle between the vectors. Note that the quantity on the left is the magnitude of the cross product, which is a scalar.

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\theta$$

A useful application in many situations is that the cross product can be used to find the area of a parallelogram defined by two vectors.



Finally, two vectors are said to be ______________________ if $\vec{v} = k\vec{w}$ for some constant k. This is typically quick to verify, but this can also be shown using the cross product.

If two vectors are parallel, then the angle between them is either $\theta = 0$ (same direction) or $\theta = \pi$ (opposite directions), which means that

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\theta = |\vec{v}|\,|\vec{w}| \cdot 0 = 0$$

Another way to think of this is that if two vectors are parallel to each other, then there are an infinite number of vectors that are parallel to both (in an infinite number of directions), meaning that they must have a cross product of $\vec{0}$ (the only vector that has more than one direction).

Example 6: Given the vectors $\vec{a} = \langle 3, 1, -1 \rangle$ and $\vec{b} = \langle 2, 2, 1 \rangle$, find $\vec{a} \times \vec{b}$ and explicitly show that $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ and $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$, meaning that the cross product is orthogonal to both $\vec{a}$ and $\vec{b}$.