University of Connecticut
Department of Mathematics

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Math 1071Q
Exam 1, Fall 2015

Duration: 120 minutes

Name: ____________________ Section: ____________________

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1. Write clearly. Points may be deducted if your work is messy or your answer unclear;
2. Answer the questions in the space provided. You may use the back of the page if necessary;
3. You must show your work or explain your solution, otherwise points may be deducted;
4. No credit will be given for incorrect steps nor will credit be given for correct solutions arrived at by incorrect means;
5. If you make an unnecessary approximation in your solution to a problem, your answer will be judged on its accuracy. Points may be deducted for poor or inappropriate approximation;

1. The revenue and cost functions for a particular product are given below. The revenue and cost are given in dollars, and x represents the number of units.

$$\text{Revenue: } R(x) = -0.2x^2 + 158x$$
$$\text{Cost: } C(x) = 62x + 11,200$$

(a) (5 points) How many items must be sold to maximize the revenue?

Solution: Revenue, $R(x)$, is a quadratic function.
Therefore the vertex point is (h, k), where

$$h = -\frac{b}{2a}$$
$$k = c - \frac{b^2}{4a}$$

and, since the coefficient of x^2 is negative, $R(x)$ will attain a maximum of k when $x = h$.
Now, $a = -0.2, b = 158$ and $c = 0$ so

$$h = -\frac{158}{2(-0.2)} = 395.$$

Thus, 395 items must be sold to maximise the profit.

(b) (2 points) What is the profit function?

Solution: The profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.2x^2 + 158x - (62x + 11,200) \\ &= -0.2x^2 + 96x - 11,200. \end{aligned}$$

(c) (5 points) At what production level(s) will the company break even?

Solution: The company will break even when $R(x) = C(x)$.
So,

$$\begin{aligned} -0.2x^2 + 158x &= 62x + 11,200 \\ -0.2x^2 + 96x - 11,200 &= 0 \end{aligned}$$

Using the quadratic formula we can get $x = 200$ or 280.

2. (a) (6 points) If a principal of \$1000 is invested in an account that earns interest at a rate of 8% per year and interest is compounded monthly, then how much will be in the account after two years?

Solution: $P = 1000$, $r = 0.08$, $m = 12$ and $t = 2$.
So,

$$F = P\left(1+\frac{r}{m}\right)^{mt} = 1000\left(1+\frac{0.08}{12}\right)^{24} \approx 1172.89.$$

(b) (6 points) How much should we invest now in order to have \$10,000 at the end of 20 years if we are investing in an account that earns interest at a rate of 9% per year and interest is compounded monthly?

Solution: $F = 10000$, $t = 20$, $r = 0.09$ and $m = 12$. So

$$P = \frac{F}{\left(1+\frac{r}{m}\right)^{mt}} = \frac{10000}{\left(1+\frac{0.09}{12}\right)^{12\cdot 20}} \approx 1664.13.$$

3. In this question, you should leave your answer in exact form.

(a) (6 points) Solve the following equation for x

$$4^{2x} = 8^{9x+15}.$$

Solution:

$$4^{2x} = 8^{9x+15}$$

so

$$(2^2)^{2x} = (2^3)^{9x+15}$$

so

$$2^{4x} = 2^{27x+45}$$

so

$$4x = 27x + 45$$

and so

$$x = -\frac{45}{23}.$$

(b) (6 points) Solve the following equation for x:

$$2 \cdot 10^{3x-1} = 1.$$

Solution:

$$2 \cdot 10^{3x-1} = 1$$

so

$$10^{3x-1} = \frac{1}{2}$$

so

$$3x - 1 = \log_{10} \frac{1}{2}$$

and so

$$x = \frac{\log_{10} \frac{1}{2} + 1}{3}.$$

4. Find all values of x where the following piecewise functions are discontinuous. *Note:* You must justify your answer.

(a) (5 points)

$$f(x) = \begin{cases} \frac{-x+1}{(x-1)(x-3)} & \text{if } x < 2 \\ \frac{x}{2} & \text{if } x \geq 2 \end{cases}.$$

Solution: Note first that

$$\frac{-x+1}{(x-1)(x-3)} = \frac{-(x-1)}{(x-1)(x-3)}$$

is not defined at $x = 1$ and $x = 3$. Therefore, $f(x)$ is discontinuous at $x = 1$. Next,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{-(x-1)}{(x-1)(x-3)} = \lim_{x \to 2^-} \frac{-1}{x-3} = 1$$

$$\text{and } \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} = 1.$$

Finally, $\frac{x}{2}$ is continuous everywhere. So, $f(x)$ is continuous everywhere except $x = 1$.

(b) (4 points)

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x < 3 \\ 2 & \text{if } x = 3 \\ \frac{1}{3}x + \frac{1}{2} & \text{if } x > 3 \end{cases}.$$

Solution:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x}{2} = \frac{3}{2}$$

$$\text{and } \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{1}{3}x + \frac{1}{2} = \frac{3}{2}.$$

However, $f(3) = 2$. So, $f(x)$ is discontinuous at $x = 3$. Moreover, $\frac{x}{2}$, 2 and $\frac{1}{3}x + \frac{1}{2}$ are continuous everywhere, so $f(x)$ is continuous except at $x = 3$.

(c) (4 points)

$$f(x) = \begin{cases} \frac{1}{3}x + \frac{1}{2} & \text{if } x \leq 6 \\ \frac{1}{36}x^2 + \frac{1}{2} & \text{if } x > 6 \end{cases}.$$

Solution: Note first that $\frac{1}{3}x + \frac{1}{2}$ and $\frac{1}{36}x^2 + \frac{1}{2}$ are continuous everywhere. Now,

$$\lim_{x \to 6^-} f(x) = \lim_{x \to 6^-} \frac{1}{3}x + \frac{1}{2} = \frac{5}{2}$$

$$\text{and } \lim_{x \to 6^+} f(x) = \lim_{x \to 6^+} \frac{1}{36}x^2 + \frac{1}{2} = \frac{3}{2}$$

so $f(x)$ is discontinuous at $x = 6$.

5. (a) (8 points) Compute the following limit or state that it does not exist:

$$\lim_{x\to 2} \frac{x^3 - 4x}{x - 2}.$$

Solution:

$$\begin{aligned}
&\lim_{x\to 2} \frac{x^3 - 4x}{x - 2} \\
&\qquad = \lim_{x\to 2} \frac{x(x^2 - 4)}{x - 2} \\
&\qquad = \lim_{x\to 2} \frac{x(x+2)(x-2)}{x - 2} \\
&\qquad = \lim_{x\to 2} x(x+2) \\
&\qquad = 2(4) = 8.
\end{aligned}$$

(b) (4 points) Find the average rate of change of the function $f(x) = 1 - \frac{2}{x}$ over the interval $[-1, 1]$.

Solution:

$$\frac{f(b) - f(a)}{b - a} = \frac{(1 - \frac{1}{1}) - (1 - \frac{2}{-1})}{2} = -2.$$

6. (a) (9 points) Use the **limit definition** of the derivative to compute $f'(x)$ where $f(x) = \frac{1}{x^2}$.
Note: No credit will be given if the limit definition is not used.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2xh - h^2}{x^2(x+h)^2}\right] \\ &= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} \\ &= \frac{-2x}{x^4} = -\frac{2}{x^3}. \end{aligned}$$

(b) (4 points) Find an equation for the tangent line to the graph of $f(x) = \frac{1}{x^2}$ at $x = 1$.

Solution: $f'(1) = -\frac{2}{1} = -2$ and $f(1) = 1$.
So, the tangent line has equation

$$y - 1 = -2(x - 1)$$

or

$$y = -2x + 3.$$

7. (a) (6 points) Find $f'(x)$ where $f(x) = x^3 - \frac{1}{\sqrt[3]{x^4}} + \pi^2$.

Solution:

$$f'(x) = 3x^2 - \left(-\frac{4}{3}\right) x^{-\frac{4}{3}-1} = 3x^2 + \frac{4}{3x^{\frac{7}{3}}}.$$

(b) (6 points) Find $f'(x)$ where $f(x) = 3e^x + \ln\left(\frac{x}{4}\right)$.

Solution: First, note that

$$f(x) = 3e^x - \ln(x) - \ln(4)$$

so

$$f'(x) = 3e^x + \frac{1}{x}.$$

8. (a) (7 points) Find $f'(x)$ where $f(x) = (e^x + 1)(\sqrt{x} + 1)$.

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^x + 1) \cdot (\sqrt{x} + 1) + (e^x + 1)\frac{d}{dx}(\sqrt{x} + 1) \\ &= (e^x)(\sqrt{x} + 1) + (e^x + 1)\left(\frac{1}{2\sqrt{x}}\right). \end{aligned}$$

(b) (7 points) Find $f'(x)$ where $f(x) = \frac{x}{x+2}$.

Solution:

$$\begin{aligned} f'(x) &= \frac{(x+2)\frac{d}{dx}(x) - x\frac{d}{dx}(x+2)}{(x+2)^2} \\ &= \frac{x + 2 - x}{(x+2)^2} = \frac{2}{(x+2)^2}. \end{aligned}$$